

Cosmological self-tuning and local solutions in generalized Horndeski theories

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Abstract

We study both the cosmological self-tuning and the local predictions (inside the Solar system) of the most general shift-symmetric beyond Horndeski theory. We first show that the cosmological self-tuning is generic in this class of theories: By adjusting a mass parameter entering the action, a large bare cosmological constant can be effectively reduced to a small observed one. Requiring then that the metric should be close enough to the Schwarzschild solution in the Solar system, to pass the experimental tests of general relativity, and taking into account the renormalization of Newton's constant, we select a subclass of models which presents all desired properties: It is able to screen a big vacuum energy density, while predicting an exact Schwarzschild-de Sitter solution around a static and spherically symmetric source. As a by-product of our study, we identify a general subclass of beyond Horndeski theory for which regular self-tuning black hole solutions exist, in presence of a time-dependent scalar field. We discuss possible future development of the present work.

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I. INTRODUCTION

The huge discrepancy of the observed value of the cosmological constant and its various theoretical predictions is a long standing problem of modern physics. The value of the energy density corresponding to the cosmological constant today, as fitted by observations using the Λ CDM model, is of order 10^{-46} GeV^4 . This value, written in units of the Planck mass (M_{Pl}) is $\sim 10^{-122}$, which is to be compared to the naive theoretical prediction of the vacuum energy of order of Planck energy density. In other words, the naive predicted value of the vacuum energy density is 10^{122} times greater than the observed one. The theoretical estimate of the value of the vacuum energy comes from the existence of a zero-point energy of the quantized fields. The zero-point energy density formally diverges, as it contains an integral over all momenta of a given energy in each mode. However, the application of a cutoff at the Planck mass gives a vacuum energy density $\rho \sim M_{\text{Pl}}^4$. It has been argued, however, that one should use a different regularization scheme, which does not break Lorentz invariance, see the review [1]. Dimensional regularization gives in particular a different answer, $|\rho| \sim 10^8 \text{ GeV}^4$ [2]. The problem is clearly alleviated, but the discrepancy remains nevertheless huge, i.e., the value of the vacuum energy density predicted in this scheme is $\sim 10^{54}$ times greater than the observed one.

Besides the above mentioned problem of zero-point energy of quantum fluctuations, there is yet another source of a big cosmological constant: phase transitions in the early Universe. In particular, the electroweak symmetry breaking, through which the gauge bosons gain their masses, is accompanied with a change of the vacuum value of the Higgs boson. This leads, in turn, to a change of vacuum energy density, which is estimated to be $|\rho_{\text{EW}}| \sim 10^8 \text{ GeV}^4$ [1]. Similar phase transition in QCD physics leads to $|\rho_{\text{QCD}}| \sim 10^{-2} \text{ GeV}^4$ [3]. Any of these predictions leads to too large vacuum energy.

Modifying gravity by the introduction of a scalar degree of freedom in the gravity sector is a promising attempt to solve the cosmological constant problem. The most general scalar-tensor theory with equations of motion up to second order in derivatives is known as the Horndeski theory [4], or, in modern formulations, the Galileons [5–7]. The absence of higher than second derivatives in the equations of motion guarantees the absence of any Ostrogradski ghost — an extra ghost degree of freedom generically associated with higher derivatives. The opposite is not always true, however, i.e., equations of motion involving higher-order derivatives do not necessary imply the appearance of an extra degree of freedom. An extension of the Horndeski theory has indeed been constructed, “beyond Horndeski” theory [8–11], which leads to third-order equations of motion, but nevertheless with only one scalar degree of freedom.¹

It has been shown that a subclass of the Horndeski/Galileon theory, called “Fab Four”, has the property of total cancellation of a bare cosmological constant [14, 15]. An extension of the Fab Four model, which includes the beyond Horndeski terms holds the same property [16]. In these scenarii the metric is flat, while the scalar field has a non-trivial configuration. Therefore this particular model cannot be realistic, since the observed Universe contains a small but non-zero cosmological constant. One should thus search for a model which would be able to *self-tune*, i.e., to naturally tune the large value of a bare cosmological constant to a small observed one. An example of such a model, in a peculiar non-linear extension of a

¹ A further extension of the beyond Horndeski theory has also been studied in [12, 13]. We however do not consider this “beyond beyond Horndeski” extension in the present paper.

subclass of Horndeski model (with an arbitrary function of the standard kinetic term plus the “John” term of the Fab Four) has been presented in [17, 18], and further studied in [19]. An approach similar to [14, 15] has been put forward in [20, 21], in order to find a subclass of the Horndeski theory which brings a bare cosmological constant down to a smaller one fixed by the theory itself.

It is however clear that a physically viable model should not only demonstrate its ability for self-tuning at the cosmological level, but it should also pass local gravity tests, in particular solar-system tests. Any considerable deviation from general relativity (GR) inside the Solar system would rule it out, in spite of its nice cosmological self-tuning. For instance, as shown in [22], a kinetic coupling between the graviton and the scalar degree of freedom leads to the appearance of an effective matter-scalar coupling (even in the case of a zero bare coupling). Such a coupling is dangerous for self-tuning models, in spite of the Vainshtein mechanism (for a review see [23]), since it may lead to a large backreaction of the scalar field. Indeed, the value of the time derivative of the scalar field is expected to be naturally large, in the self-tuning scenario. At the same time, as it has been shown for the cubic Galileon model, the induced matter-scalar coupling is proportional to this time derivative of the scalar field [22]. Therefore one may expect that the backreaction of the scalar onto the geometry is large, so that solar-system tests are not passed.

It is therefore important to identify the models which produce self-tuning to a small observed cosmological constant, but at the same time do not spoil solar-system tests. One such example has been studied in [24] (see also [25, 26]): a model containing the “John” term of the Fab Four not only provides an asymptotically de Sitter spacetime with an effective cosmological constant, independent from the bare vacuum constant, but it also gives a GR-like solution near a central source.

In this paper, we systematically study cosmology and local behavior of all shift-symmetric generalized Galileon (beyond Horndeski) models. The action we consider, defined in Sec. II, contains six arbitrary functions of the standard kinetic term of a scalar field.² The Horndeski theory corresponds to a particular choice of two of these functions in terms of the other four, thus reducing the space of the general model to four arbitrary functions. We provide in Sec. II a translation of our action in terms of other notations which have been used in the literature. We also show that the Einstein equations can be significantly simplified when they are combined with the scalar current (whose divergence gives the scalar-field equation).

In the first part of the paper (Sec. III), we focus on homogeneous cosmology of beyond Horndeski models. We derive their general field equations, and use them to discuss several illustrative examples of self-tuning. We show, in particular, that an extra scale in the action (besides the Planck scale) is necessary for the model to exhibit self-tuning. This scale however does not need to be of order of the Hubble scale today, although such a scenario is also allowed. Moreover, the value of the time derivative of the scalar field can also be adjusted to have either large or small values, depending on the theory.

In the second part (Sec. IV), we select a subset of the beyond Horndeski theories, which provides self-tuning mechanism in cosmology *and* also restores the GR behavior for the metric around a central source. More precisely, we exhibit a subclass of models, depending on six functions of the standard kinetic term, admitting an exact Schwarzschild-de Sitter

² Note that not all combinations of beyond Horndeski theories are free from the Ostrogradski ghost. One cannot mix simultaneously Horndeski and beyond Horndeski \mathcal{L}_4 and \mathcal{L}_5 terms; see Refs. [12, 27] and a short summary in Sec. II below.

solution around a spherical mass, and therefore *a priori* able to pass local gravity tests. Three of these functions (that we call the “Three Graces”) contribute actively to the self-tuning solution, i.e., to the fact that the observed cosmological constant Λ_{eff} is much smaller than the bare one Λ_{bare} entering the action. These functions need to be related to each other in a specific way for the model to admit a Schwarzschild-de Sitter solution. The three extra functions, which also need to satisfy some relations, correspond to “stealth” Lagrangians: They are allowed and do contribute to the physics of perturbations, but they do not affect neither the background cosmological solution nor the local spherically symmetric metric. In Sec. IV B, we show that these exact Schwarzschild-de Sitter solutions also describe regular black holes, generalizing the self-tuning black hole solutions obtained in [24] and extended in [28].

In Sec. V, we adopt a perturbative approach to study which models can predict a metric close enough to the Schwarzschild solution to pass solar-system tests, while not giving the *exact* Schwarzschild-de Sitter solutions of Sec. IV. We also use our perturbed field equations to show that the observed Newton’s constant G is generically renormalized with respect to the bare one entering the action of the theory, notably in the Three Graces of Sec. IV. This causes the cosmological constant problem *not* to be solved in most of the cases, but we show that a subclass of models presents all desired properties: It predicts a tiny observed cosmological constant, an exact Schwarzschild-de Sitter metric around a central source, and no renormalization of Newton’s constant, so that the observed vacuum energy density can be negligible with respect to the bare one entering the action.

We finally give our conclusions in Sec. VI.

II. GENERALIZED HORNDESKI THEORIES

Generalized Galileon Lagrangians are most conveniently written as contractions with two (fully-antisymmetric) Levi-Civita tensors [29, 30]. Their main property is then manifest, namely that their field equations in flat spacetime depend only on second derivatives of the scalar field φ .

In four dimensions, there are six possible such Lagrangians. We quote below their simplest definitions, followed by their much heavier expansions in terms of contracted covariant derivatives and curvature tensors. We use the sign conventions of [31], notably the mostly-plus signature, and denote as $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ the Einstein tensor. We also denote as $\varphi_{\mu} \equiv \partial_{\mu}\varphi$ the first derivative of the scalar field, and similarly as $\varphi_{\mu\nu} \equiv \nabla_{\nu}\nabla_{\mu}\varphi$ its second

covariant derivative. The six generalized Galileon Lagrangians read

$$L_{(2,0)} \equiv -\frac{1}{3!} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^\alpha_{\nu\rho\sigma} \varphi_\mu \varphi_\alpha = (\partial_\mu \varphi)^2, \quad (1a)$$

$$L_{(3,0)} \equiv -\frac{1}{2!} \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta}_{\rho\sigma} \varphi_\mu \varphi_\alpha \varphi_{\nu\beta} = (\varphi_\mu)^2 \square \varphi - \varphi^\mu \varphi_{\mu\nu} \varphi^\nu, \quad (1b)$$

$$L_{(4,0)} \equiv -\varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma}_\sigma \varphi_\mu \varphi_\alpha \varphi_{\nu\beta} \varphi_{\rho\gamma} \quad (1c)$$

$$= (\varphi_\mu)^2 (\square \varphi)^2 - 2 \varphi^\mu \varphi_{\mu\nu} \varphi^\nu \square \varphi - (\varphi_\mu)^2 (\varphi_{\nu\rho})^2 + 2 \varphi^\mu \varphi_{\mu\nu} \varphi^{\nu\rho} \varphi_\rho, \quad (1d)$$

$$L_{(5,0)} \equiv -\varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \varphi_\mu \varphi_\alpha \varphi_{\nu\beta} \varphi_{\rho\gamma} \varphi_{\sigma\delta} \quad (1e)$$

$$\begin{aligned} &= (\varphi_\mu)^2 (\square \varphi)^3 - 3 (\varphi^\mu \varphi_{\mu\nu} \varphi^\nu) (\square \varphi)^2 - 3 (\varphi_\mu)^2 (\varphi_{\nu\rho})^2 \square \varphi \\ &\quad + 6 \varphi^\mu \varphi_{\mu\nu} \varphi^{\nu\rho} \varphi_\rho \square \varphi + 2 (\varphi_\mu)^2 \varphi_{\nu}{}^\rho \varphi_\rho{}^\sigma \varphi_\sigma{}^\nu \\ &\quad + 3 (\varphi_{\mu\nu})^2 \varphi^\rho \varphi_{\rho\sigma} \varphi^\sigma - 6 \varphi^\mu \varphi_{\mu\nu} \varphi^{\nu\rho} \varphi_{\rho\sigma} \varphi^\sigma, \end{aligned} \quad (1f)$$

$$L_{(4,1)} \equiv -\varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma}_\sigma \varphi_\mu \varphi_\alpha R_{\nu\rho\beta\gamma} = -4 G^{\mu\nu} \varphi_\mu \varphi_\nu, \quad (1g)$$

$$L_{(5,1)} \equiv -\varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \varphi_\mu \varphi_\alpha \varphi_{\nu\beta} R_{\rho\sigma\gamma\delta} \quad (1h)$$

$$\begin{aligned} &= 2 (\varphi_\mu)^2 R \square \varphi - 2 \varphi^\mu \varphi_{\mu\nu} \varphi^\nu R - 4 \varphi^\mu R_{\mu\nu} \varphi^\nu \square \varphi \\ &\quad - 4 (\varphi_\mu)^2 \varphi^{\nu\rho} R_{\nu\rho} + 8 \varphi^\mu \varphi_{\mu\nu} R^{\nu\rho} \varphi_\rho + 4 \varphi^\mu \varphi^\nu \varphi^{\rho\sigma} R_{\mu\rho\nu\sigma}. \end{aligned} \quad (1i)$$

These definitions coincide with those of [11, 29] for all $L_{(n,0)}$. For those involving one Riemann tensor, $L_{(n,1)}$, we decided to simplify them by removing a factor $(\varphi_\lambda)^2$. We shall indeed multiply below all these Lagrangians by arbitrary functions of $(\varphi_\lambda)^2$, therefore this extra factor was unnecessarily heavy in definitions (1g) and (1h). Note that when multiplying the above Lagrangians (1g) and (1h) by arbitrary functions of φ , $L_{(4,1)}$ was nicknamed “John” in the “Fab Four” model [14, 15], while $L_{(5,1)}$ was nicknamed “Paul”.

Generalized Horndeski theories correspond to multiplying the above Lagrangians by arbitrary functions of both the scalar field φ and its standard kinetic term $(\varphi_\lambda)^2$. [We shall recall the difference between Horndeski and generalized Horndeski theories below Eqs. (6).] In the present paper, we will focus on shift-symmetric theories, whose actions do not involve any undifferentiated φ , and we shall thus only multiply the above Lagrangians by functions of $(\varphi_\lambda)^2$.

In the following, we choose that φ is dimensionless, but introduce a mass scale M so that all Lagrangians have the same dimension. The functions will depend on the dimensionless ratio

$$X \equiv \frac{-(\varphi_\lambda)^2}{M^2}. \quad (2)$$

Note the sign, the absence of a factor $\frac{1}{2}$, and the $1/M^2$ factor, as the notation X is used with various definitions in the literature. Our negative sign is chosen so that $X > 0$ in cosmological situations, where the time derivative $\dot{\varphi}$ of the scalar field is dominating over its spatial derivatives.

In addition to the mass scale M , which will be the only one we use in the scalar field kinetic terms, the action we consider also depends on two other scales: the reduced Planck mass $M_{\text{Pl}} \equiv (8\pi G)^{-1/2}$ (in units such that $\hbar = c = 1$), which multiplies the Einstein-Hilbert action, and a bare cosmological constant Λ_{bare} , which may be much larger than the observed one (see Sec. III below for a discussion of the effective cosmological constant Λ_{eff} which is actually observed). A simple framework would be for instance to assume that $\Lambda_{\text{bare}} = \mathcal{O}(M_{\text{Pl}}^2)$, so that the model depend only on two scales, M and M_{Pl} . Let us stress that that the measured Newton’s constant, for instance in Cavendish experiments, is not

the bare $G \equiv 8\pi/M_{\text{Pl}}^2$ we introduce in this action, but it acquires a renormalized value G_{eff} . Our notation M_{Pl} and G should thus be understood as bare parameters, whose numerical values are not known yet. We will relate them to the observed ones in Sec. V, for a specific class of models which reproduces the Schwarzschild solution in the vicinity of a spherical body. In order not to introduce extra hidden scales in the model, we will assume that all functions of X defined below involve dimensionless coefficients of order $\mathcal{O}(1)$.

The class of theories we are considering is thus defined by the full action

$$S = \frac{M_{\text{Pl}}^2}{2} \int \sqrt{-g} (R - 2\Lambda_{\text{bare}}) d^4x + \sum_{(n,p)} \int \sqrt{-g} \mathcal{L}_{(n,p)} d^4x + S_{\text{matter}}[\psi, g_{\mu\nu}], \quad (3)$$

where all matter fields different from φ (globally denoted as ψ) are assumed to be universally coupled to $g_{\mu\nu}$ but not directly to φ , and where the generalized Horndeski Lagrangians $\mathcal{L}_{(n,p)}$ are related to the generalized Galileon ones (1) by

$$\mathcal{L}_{(2,0)} = M^2 f_2(X) L_{(2,0)} = -M^4 X f_2(X), \quad (4a)$$

$$\mathcal{L}_{(3,0)} = f_3(X) L_{(3,0)}, \quad (4b)$$

$$\mathcal{L}_{(4,0)} = \frac{1}{M^2} f_4(X) L_{(4,0)}, \quad (4c)$$

$$\mathcal{L}_{(5,0)} = \frac{1}{M^4} f_5(X) L_{(5,0)}, \quad (4d)$$

$$\mathcal{L}_{(4,1)} = s_4(X) L_{(4,1)}, \quad (4e)$$

$$\mathcal{L}_{(5,1)} = \frac{1}{M^2} s_5(X) L_{(5,1)}. \quad (4f)$$

Since different notation is used in the literature to define these theories, let us give a dictionary. First of all, let us recall that the $L_{(4,1)}$ and $L_{(5,1)}$ of Refs. [11, 29] were not defined as in Eqs. (1g) and (1h) above, but rather as $\mathcal{L}_{(4,1)}$ and $\mathcal{L}_{(5,1)}$, Eqs. (4e) and (4f), with $s_4 = s_5 = -X$ and $M = 1$. Second, generalized Horndeski theories were first defined in [8, 9, 12] with a notation mixing the $G_n(\varphi_\lambda^2)$ functions used for the Horndeski theory [4, 7, 32] and new functions $F_n(\varphi_\lambda^2)$ multiplying the above contractions (1c) and (1e) with two Levi-Civita tensors:

$$\mathcal{L}_{(2,0)} = G_2(\varphi_\lambda^2), \quad (5a)$$

$$\mathcal{L}_{(3,0)} = G_3(\varphi_\lambda^2) \square\varphi + \text{tot. div.}, \quad (5b)$$

$$\begin{aligned} \mathcal{L}_{(4,0)} + \mathcal{L}_{(4,1)} = & G_4(\varphi_\lambda^2) R - 2G'_4(\varphi_\lambda^2) \left[(\square\varphi)^2 - \varphi_{\mu\nu} \varphi^{\mu\nu} \right] \\ & + F_4(\varphi_\lambda^2) \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma}{}_\sigma \varphi_\mu \varphi_\alpha \varphi_{\nu\beta} \varphi_{\rho\gamma} + \text{tot. div.}, \end{aligned} \quad (5c)$$

$$\begin{aligned} \mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)} = & G_5(\varphi_\lambda^2) G^{\mu\nu} \varphi_{\mu\nu} + \frac{1}{3} G'_5(\varphi_\lambda^2) \left[(\square\varphi)^3 - 3 \square\varphi \varphi_{\mu\nu} \varphi^{\mu\nu} + 2 \varphi_{\mu\nu} \varphi^{\nu\rho} \varphi_\rho{}^\mu \right] \\ & + F_5(\varphi_\lambda^2) \varepsilon^{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} \varphi_\mu \varphi_\alpha \varphi_{\nu\beta} \varphi_{\rho\gamma} \varphi_{\sigma\delta} + \text{tot. div.}, \end{aligned} \quad (5d)$$

where $G'_4(\varphi_\lambda^2)$ and $G'_5(\varphi_\lambda^2)$ mean the derivatives of these functions with respect to their argument, i.e., $G'_n(\varphi_\lambda^2) = dG_n(\varphi_\lambda^2)/d(\varphi_\lambda^2) = dG_n(-M^2 X)/d(-M^2 X)$. The partial integrations given in Appendix A below imply that these functions G_n and F_n are related to our

f_n and s_n as follows:

$$G_2(-M^2 X) = -M^4 X f_2(X), \quad (6a)$$

$$G_3(-M^2 X) = -M^2 \left[X f_3(X) + \frac{1}{2} \int f_3(X) dX \right], \quad (6b)$$

$$G_4(-M^2 X) = -2M^2 X s_4(X), \quad (6c)$$

$$F_4(-M^2 X) = [-f_4(X) + 4s'_4(X)]/M^2, \quad (6d)$$

$$G_5(-M^2 X) = 4X s_5(X) + 2 \int s_5(X) dX, \quad (6e)$$

$$F_5(-M^2 X) = \left[-f_5(X) + \frac{4}{3} s'_5(X) \right] / M^4. \quad (6f)$$

When the functions $F_{4,5} = 0$, Horndeski [4] showed that the field equations of these models involve at most second derivatives. When these extra functions $F_{4,5}$ are also present (or when our $f_{4,5}$ and $s_{4,5}$ of Eq. (4) are independent), then third derivatives of the metric $g_{\mu\nu}$ appear in the scalar-field equation, and third derivatives of φ in the Einstein equations. One could thus fear that this is associated with an extra degree of freedom, which is generally a ghost [33], and that the models are then unstable. References [8–10] underlined that this is not the case. Their initial arguments were actually inconclusive, notably because they were working in a specific gauge where extra degrees of freedom may be hidden (the reason being their choice of a non-generic initial value surface corresponding to $\varphi = \text{const.}$). However, a full-fledged Hamiltonian analysis of the particular case of $L_{(4,0)}$, Eq. (1c), without fixing any gauge, did show that there is indeed no extra degree of freedom [11]. Under the simplifying but reasonable hypothesis that the spin-2 sector does not hide any subtlety, Refs. [12, 13, 27] then showed that this is also the case for *most* of these generalized Horndeski models, but curiously enough, not all of them — confirming thereby that previously published arguments were incomplete. It was shown in [12, 27] that any combination of $\mathcal{L}_{(4,0)}$ and $\mathcal{L}_{(4,1)}$, Eqs. (4c) and (4e), or equivalently of the functions F_4 and G_4 , is free of any extra degree of freedom, and this results also holds for any combination of $\mathcal{L}_{(5,0)}$ and $\mathcal{L}_{(5,1)}$, Eqs. (4d) and (4f). This is still the case when combining any G_4 with any G_5 , but with $F_4 = F_5 = 0$ (i.e., within the class of Horndeski theories [4]), as well as when combining any F_4 with any F_5 , but with $G_4 = G_5 = 0$ (i.e., when considering the Lagrangian $\mathcal{L}_{(4,0)} + \mathcal{L}_{(5,0)}$ without their curvature-dependent counterparts $\mathcal{L}_{(4,1)}$ nor $\mathcal{L}_{(5,1)}$). On the other hand, there does generically exist an extra degree of freedom when combining arbitrary $\mathcal{L}_{(4,0)}$, $\mathcal{L}_{(4,1)}$, $\mathcal{L}_{(5,0)}$ and $\mathcal{L}_{(5,1)}$. In the following, we shall study the most general case, but one should keep in mind that all four functions f_4 , s_4 , f_5 and s_5 together usually correspond to unstable models.

When performing their Hamiltonian analysis in the unitary gauge, Refs. [8–10] introduced yet another notation, with functions A_n and B_n ; see Eqs. (25)–(29) and Appendix A of Ref. [9]. Our Appendix A below or Eqs. (6) allow us to relate them to our f_n and s_n as follows:

$$A_2(-M^2 X) = -M^4 X f_2(X), \quad (7a)$$

$$A_3(-M^2 X) = M^3 X^{3/2} f_3(X), \quad (7b)$$

$$A_4(-M^2 X) = -M^2 X [X f_4(X) + 2s_4(X)], \quad (7c)$$

$$B_4(-M^2 X) = -2M^2 X s_4(X), \quad (7d)$$

$$A_5(-M^2 X) = M X^{3/2} [X f_5(X) + 2s_5(X)], \quad (7e)$$

$$B_5(-M^2 X) = -4M X^{3/2} s_5(X). \quad (7f)$$

The field equations are obtained by varying action (3) with respect to the metric and the scalar field. However, since we restrict to theories which do not depend on any undifferentiated φ , we may write the scalar field equation as the conservation of a current. We define the energy-momentum tensor of the scalar field and the scalar current respectively as

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S[\varphi]}{\delta g_{\mu\nu}}, \quad (8a)$$

$$J^\mu \equiv \frac{-1}{\sqrt{-g}} \frac{\delta S[\varphi]}{\delta(\partial_\mu \varphi)}, \quad (8b)$$

where $S[\varphi] = \sum \int \sqrt{-g} \mathcal{L}_{(n,p)}$ is the φ -dependent part of action (3), and the field equations read thus

$$G_{\mu\nu} + \Lambda_{\text{bare}} g_{\mu\nu} = \frac{T_{\mu\nu}}{M_{\text{Pl}}^2}, \quad (9a)$$

$$\nabla_\mu J^\mu = 0, \quad (9b)$$

where we do not write the matter contribution to Eq. (9a) because we will solve these field equations in the exterior of material bodies. This matter contribution will anyway be useful to fix the constant of integration of Einstein's equations.

It should be underlined that the solutions to Eq. (9b) do not correspond to $J^\mu = 0$ in general. For instance, if χ is any solution of the free propagation equation $\square\chi = 0$ in a given curved background, it is clear that $\partial^\mu\chi$ may be added to J^μ without changing Eq. (9b). However, in the very symmetric situations we will consider below (homogeneous and isotropic Universe in Sec. III, and static and spherically symmetric solution in Sec. IV), we will see that the precise values of some components of J^μ may be determined, and our solutions to (9b) will actually correspond to imposing $J^\mu = 0$ for some specific index $\underline{\mu}$. It happens that the Einstein equations (9a) can be simplified a lot by combining them with the current as

$$G^{\mu\nu} + \Lambda_{\text{bare}} g^{\mu\nu} - \frac{T^{\mu\nu}}{M_{\text{Pl}}^2} + \frac{J^\mu \varphi^\nu}{M_{\text{Pl}}^2}. \quad (10)$$

Indeed, most of the terms involving derivatives of the functions $f_n(X)$ and $s_n(X)$ cancel in such a combination. To prove so, a quick and naive argument is to note that if some f' or s' were involved in (10), then some f'' or s'' should be generated when taking the divergence of (10). But diffeomorphism invariance of action (3) implies that

$$\nabla_\mu \left(G^{\mu\nu} + \Lambda_{\text{bare}} g^{\mu\nu} - \frac{T^{\mu\nu}}{M_{\text{Pl}}^2} \right) = -\frac{\varphi^\nu \nabla_\mu J^\mu}{M_{\text{Pl}}^2}. \quad (11)$$

Therefore, the divergence of (10) is just equal to $J^\mu \nabla_\mu \varphi^\nu / M_{\text{Pl}}^2$, which cannot contain any second derivative f'' nor s'' . This argument is however incomplete, because $T^{\mu\nu}$ also contains “superpotential” terms

$$4\nabla_{(\rho} \nabla_{\sigma)} \left\{ \varepsilon^{\mu\rho\alpha\gamma} \varepsilon^{\nu\sigma\beta\delta} \varphi_\alpha \varphi_\beta [g_{\gamma\delta} s_4(X) + \varphi_{\gamma\delta} s_5(X)/M^2] \right\}, \quad (12)$$

coming from the variation of the Riemann tensor in (1g) and (1h) with respect to the metric. Such terms automatically vanish when taking a divergence, because of the full antisymmetry

of the two Levi-Civita tensors. Therefore, although they contain in general first and even second derivatives of the functions s_4 and s_5 , they do not contribute to the divergence of (10). The conclusion is that the above combination (10) of the Einstein equations with the scalar current cancels *most* of the first derivatives of the functions $f_n(X)$ and $s_n(X)$, but not all of them. We will use it in Secs. III and IV below, and we will see that such first derivatives actually do cancel in two important cases. In the homogenous and isotropic case of Sec. III, the reason is that we will focus on the time-time component of the Einstein equations, i.e., $\mu = \nu = 0$ in Eq. (12). But in order to create first derivatives of s_4 or s_5 , at least one of the covariant derivatives ∇_ρ or ∇_σ must act on these functions, therefore at least one among φ_α or φ_β must not be differentiated any more, and should thus correspond to the only nonvanishing component $\varphi_0 = \dot{\varphi}$ in this cosmological background. In other words, one must have $\alpha = 0$ or/and $\beta = 0$ to create a derivative of s in Eq. (12), and there will thus be two indices 0 contracted with the same antisymmetric tensor ε , either $\mu = \alpha = 0$ or/and $\nu = \beta = 0$. This explains why (12) will not contribute to T^{00} in Sec. III below, and why all derivatives of $f_n(X)$ or $s_n(X)$ will be canceled in the combination (10). In the static and spherically symmetric case of Sec. IV, we will see that some s'_4 and s'_5 do remain, but they cancel in the particular case $X = \text{const.}$ that we will study (and they actually cancel as soon as one assumes $X = \text{const.}$, independently of spherical symmetry). It is indeed clear that (12) does not contribute to any derivative of s_4 nor s_5 if $X = \text{const.}$ It is also easy to prove that the only derivatives of functions f or s entering $T^{\mu\nu}$ are of the form $2(\varphi^\mu\varphi^\nu/M^2)f'(X)L_{(n,p)}$ when $X = \text{const.}$ (or with s' instead of f'), because they come from the variation of the metric used in the contraction $X = -g^{\mu\nu}\varphi_\mu\varphi_\nu/M^2$. On the other hand, the scalar current J^μ contains $2(\varphi^\mu/M^2)f'(X)L_{(n,p)}$, and no other derivative of a function when we assume $X = \text{const.}$ Therefore, the linear combination (10) obviously cancels the few possible f' or s' which occur when $X = \text{const.}$

III. COSMOLOGICAL SELF-TUNING

We consider a homogeneous and isotropic Universe whose metric takes the Friedmann-Lemaître-Robertson-Walker (FLRW) form

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (13)$$

the parameter $k \in \{-1, 0, 1\}$ determining whether the spatial hypersurfaces are open, flat or closed, and we assume consistently that the scalar field φ depends only on t . Then its current equation (9b) simply reads $\nabla_0 J^0 = 0 \Rightarrow \partial_t(a^3 J^0) = 0$, and its solution is thus $J^0 = C_0/a^3$, where C_0 is a constant. This integration constant may be neglected at late enough times, when the scale factor a becomes very large, and we will thus just solve for $J^0 = 0$ in the following (keeping in mind that an extra C_0/a^3 may be added to it).

Once the matter field equations are taken into account, i.e., $\nabla_\mu J^\mu = 0$ in the present case, it is well known that only the time-time-component of the Einstein equations (9a) needs to be solved. Indeed, the covariant conservation of the Einstein tensor $\nabla_\mu G^{\mu\nu} = 0$ implies $G^{ij} = -g^{ij} [G^{00} + \dot{G}^{00}/(3H)]$, where a dot denotes time differentiation and $H \equiv \dot{a}/a$, therefore the spatial components of the Einstein tensor are automatically solved once G^{00} is, while all off-diagonal components vanish identically. When taking into account the energy-momentum of the scalar field, this relation remains valid for $G^{\mu\nu} - T^{\mu\nu}/M_{\text{Pl}}^2$ instead of $G^{\mu\nu}$,

up to terms proportional to the scalar equation $\nabla_\mu J^\mu = 0$, because of Eq. (11) implied by the diffeomorphism invariance of action (3).

We thus only write below the 00-component of the Einstein equations and the scalar current. Actually, instead of the Einstein equation, we choose to write the 00-component of the linear combination (10), which simplifies significantly its expression, and we then display $J^0/(2M^2\dot{\varphi})$:

$$3 \left(H^2 + \frac{k}{a^2} \right) - \frac{M^4}{M_{\text{Pl}}^2} X f_2 + 6H^2 \frac{M^2}{M_{\text{Pl}}^2} X^2 f_4 + 12 \left(H^2 - \frac{k}{a^2} \right) \frac{M^2}{M_{\text{Pl}}^2} X s_4 - 12 \frac{H^3 M}{M_{\text{Pl}}^2} [X^{5/2} f_5 + 2X^{3/2} s_5] = \Lambda_{\text{bare}}, \quad (14a)$$

$$[X f_2]' - 3 \frac{H}{M} [X^{3/2} f_3]' + 6 \left(\frac{H}{M} \right)^2 [X^2 f_4]' - 6 \left(\frac{H}{M} \right)^3 [X^{5/2} f_5]' + 6 \frac{(H^2 + k/a^2)}{M^2} [2X s_4]' - 12 \frac{H(H^2 + k/a^2)}{M^3} [X^{3/2} s_5]' = 0, \quad (14b)$$

where a prime denotes differentiation with respect to X . [If the integration constant of $J^0 = C_0/a^3$ is not neglected, then this adds $C_0\dot{\varphi}/(a^3 M_{\text{Pl}}^2)$ to the right-hand side (r.h.s.) of Eq. (14a) and $C_0/(2a^3 M^2 \dot{\varphi})$ to the r.h.s. of Eq. (14b).] Note the different signs of the spatial curvature contribution k/a^2 in the various terms. This comes from the fact that it does not enter with the same weight in the Einstein equation and the scalar current. For instance, this curvature contribution happens to vanish for the s_5 term in the linear combination (14a), whereas Eq. (14b) shows that it was initially present both in T^{00} and J^0 .

Note also that we do *not* assume $\dot{H} = 0$ nor $\dot{\varphi} = 0$ in these equations: They are fully general in FLRW. The third usual cosmological equation, involving \dot{H} , is a consequence of the above two. However, the solutions we will exhibit below will actually correspond to a de Sitter Universe with $3H^2 = \Lambda_{\text{eff}} = \text{const.}$ and $\dot{\varphi} = \text{const.}$

We checked Eqs. (14) by two independent methods: First by deriving the full covariant equations and specifying them to metric (13); and second the “minisuperspace” technique, in which the form (13) is imposed directly within the action (with an arbitrary $g_{00}(t)dt^2$ instead of $-dt^2$), and varying it with respect to the two fields $g_{00}(t)$ and $\dot{\varphi}(t)$. [The non-independent rr -component of the Einstein equations is also immediate to obtain by varying this action with respect to $a(t)$.]

In the following, we shall assume that the spatial curvature vanishes, $k = 0$. In such a case, Eq. (14) simplifies even more, and our two cosmological equations take elegant similar forms:

$$-X f_2 + 6 \left(\frac{H}{M} \right)^2 [X^2 f_4 + 2X s_4] - 12 \left(\frac{H}{M} \right)^3 [X^{5/2} f_5 + 2X^{3/2} s_5] = \frac{M_{\text{Pl}}^2}{M^4} (\Lambda_{\text{bare}} - 3H^2), \quad (15a)$$

$$[X f_2]' - 3 \frac{H}{M} [X^{3/2} f_3]' + 6 \left(\frac{H}{M} \right)^2 [X^2 f_4 + 2X s_4]' - 6 \left(\frac{H}{M} \right)^3 [X^{5/2} f_5 + 2X^{3/2} s_5]' = 0. \quad (15b)$$

Note that the combinations of functions entering the various square brackets are precisely those which correspond to the A_n notation of Eq. (7). This comes from the fact that this notation was introduced while performing an ADM decomposition in the unitary gauge, which is close to the above factorization of powers of H/M , in this cosmological context where all fields only depend on time.

This writing immediately exhibits some singular limiting cases, in the present cosmological framework. First, any linear combination of $f_2 \propto 1/X$, $f_3 \propto X^{-3/2}$, $f_4 \propto X^{-2}$, $f_5 \propto X^{-5/2}$, $s_4 \propto 1/X$, and $s_5 \propto X^{-3/2}$, obviously gives a trivially satisfied $0 = 0$ equation for the current (15b), while the Einstein equation (15a) becomes fully independent from X . Therefore, such models fail at predicting which cosmological value X should take (let us call it X_c), and $\dot{\varphi}_c = M\sqrt{X_c}$ is thus free. On the other hand, such models anyway do predict a specific value of H (when the signs of the various terms are consistent), and thereby of the observed cosmological constant $\Lambda_{\text{eff}} = 3H^2$. It should however be noted that two Lagrangians among those are particularly trivial. The first one is $\mathcal{L}_{(2,0)}$ when $f_2 \propto 1/X$. Then Eq. (4a) shows that this Lagrangian is nothing else than a second bare cosmological constant — which can obviously almost compensate Λ_{bare} , but this would be equivalent to assuming that there is no large bare cosmological constant in our initial action (3). The second trivial case is when $f_3 \propto X^{-3/2}$ in the Lagrangian $\mathcal{L}_{(3,0)}$, Eq. (4b). Then it happens to be a total derivative, and thereby not to contribute to any field equation, even in generic non-symmetric situations. We have indeed $-\nabla_\mu \left[\varphi^\mu / \sqrt{-\varphi_\lambda^2} \right] = (-\varphi_\lambda^2)^{-3/2} [\varphi_\mu^2 \square\varphi - \varphi^\mu \varphi_{\mu\nu} \varphi^\nu] = \mathcal{L}_{(3,0)}/M^3$. Aside from these two trivial cases, the other $f_4 \propto X^{-2}$, $f_5 \propto X^{-5/2}$, $s_4 \propto 1/X$ and $s_5 \propto X^{-3/2}$ are perfectly allowed, notably when they are combined with other terms which are not of this specific form. One should just keep in mind that these limiting cases do not contribute to the current (15b), and this constrains the form of the other terms added to them. For instance, if one considers the sum of $\mathcal{L}_{(2,0)}$ with one of these limiting cases, then Eq. (15b) implies that one must have $(Xf_2)' = 0$, i.e., that $\mathcal{L}_{(2,0)}$ behaves a second cosmological constant at least around the background value $X = X_c$, and this could not be called an actual “self-tuning”.

Another class of non-fully predictive models is also exhibited by Eqs. (15). When f_3 , f_4 , f_5 , s_4 and s_5 are monomials (some of them possibly vanishing) such that

$$(Xf_3)^6 \propto (Xf_4 + 2s_4)^3 \propto (Xf_5 + 2s_5)^2, \quad (16)$$

and $f_2 = 0$, or when

$$X^2 f_4 + 2X s_4 = \text{const.} \quad \text{or/and} \quad X^{5/2} f_5 + 2X^{3/2} s_5 = \text{const.}, \quad (17)$$

and $f_2 = f_3 = 0$ (or the trivial $f_2 \propto 1/X$ or $f_3 \propto X^{-3/2}$ mentioned above), then the two equations depend on only one variable, which is the product of H by a given function of X . This variable is $HX^{3/2}f_3$ in the case of Eq. (16) [or $H(X^2 f_4 + 2X s_4)^{1/2} \propto H(X^{5/2} f_5 + 2X^{3/2} s_5)^{1/3}$ if $f_3 = 0$], and simply H in the case of Eq. (17). Therefore, H and X cannot be both predicted in these particular models. Although this *a priori* means that some kinetic term vanishes, this does not necessarily rule out such models. Indeed, their Cauchy problem may be well posed around slightly different backgrounds, and their non-predictivity may thus happen only when assuming an exact FLRW metric (13). It remains that we must disregard them in the present FLRW framework, since they let undetermined a combination of H and $X = (\dot{\varphi}/M)^2$. Note that in the case of Eq. (17), the current equation (15b) is trivially satisfied, but not in the case of Eq. (16), for which (15) become now two independent equations of a single variable. To make them consistent with each other, it is thus necessary to impose a very specific value of M , depending on the other parameters of the model.

Coming back now to the *generic* case of Eqs. (15), we find that self-tuning is possible with *any* combination of at least two Lagrangians $\mathcal{L}_{(n,p)}$, Eqs. (4). Indeed, although the physical meaning of these equations is to predict H and X in a given theory [defined by fixed values of M , M_{Pl} , Λ_{bare} and fixed functions $f_n(X)$ and $s_n(X)$], we may also consider them as equations determining M and X in terms of the *observed* value of H and the other parameters and functions defining the model. Therefore, it suffices to tune the mass scale M to an appropriate numerical value to get the observed $3H^2 = \Lambda_{\text{eff}}$ as a solution, in spite of the large bare cosmological constant which is introduced in action (3). Of course, the fact that we need to tune M means that we are actually introducing by hand some information about H in our action. However, this parameter M defines the dynamics of the scalar field φ , and has thus no relation with the quantum vacuum energy density. It is therefore less problematic to tune its value. Moreover, we will see that M does not need to be of the same order of magnitude as H : There exist self-tuning models with all kinds of values for M , e.g., $M \ll H$, $M \approx H$, $M \gg H$, and even trans-Planckian $M \gg M_{\text{Pl}}$ [but note that this scale M is not the energy of any localized wave packet, nor even a mass, but simply a dimensionful scale necessary to define the Lagrangians (4)]. It is notably possible to have M of an intermediate magnitude, larger than the heaviest masses of the Standard Model of particle physics but smaller than the Planck mass.

The only order of magnitude that M cannot consistently take is $M \approx (M_{\text{Pl}}^2 \Lambda_{\text{bare}})^{1/4}$. Indeed, if this were the case, then Eqs. (15) would involve only this scale and dimensionless numbers assumed to be of order 1, therefore the predicted H would also be generically of this order, i.e., much larger than the observed one. Actually, if one enforces $M = \mathcal{O}(M_{\text{Pl}}^2 \Lambda_{\text{bare}})^{1/4}$ in Eqs. (15), one can find models which would still be consistent with a small H , but they are in the “non-fully predictive” class mentioned in Eq. (16) above. They indeed predict the value of a product of H with a function of X , but nothing else. Therefore, if it happens that this function of X takes a large value in our Universe, then this will correspond to a small observed H , but any other value of H would have also been possible. In such non-predictive models, the absence of a second scale in the action is thus compensated by the random scale that can take the function of X . Let us just quote one example in this non-fully predictive class. If $f_3 = 1$, $f_4 = X$ and all other functions vanish, then both field equations (15) depend only on the product $HX^{3/2}$, and they are consistent with each other only if one imposes $M^4 = \frac{8}{3} M_{\text{Pl}}^2 \Lambda_{\text{bare}} = \mathcal{O}(M_{\text{Pl}}^2 \Lambda_{\text{bare}})$. Then they predict $HX^{3/2} = \frac{1}{4} M$, so that H may be small if X happens to be large, but a large value of H is equally allowed by the same equations. Although amusing, we shall disregard such single-scale non-fully predictive models in the following. Those we will focus on will therefore necessarily involve a second scale M , either large or small with respect to the vacuum energy scale $(M_{\text{Pl}}^2 \Lambda_{\text{bare}})^{1/4}$.

As mentioned above, self-tuning is possible when at least two Lagrangians $\mathcal{L}_{(n,p)}$ are present in the theory, and we did study systematically all possible combinations of them. Let us just quote here some examples, to illustrate the diversity of their predictions. We shall see in Secs. IV and VB that the subclass of models $\mathcal{L}_{(2,0)} + \mathcal{L}_{(4,0)} + \mathcal{L}_{(4,1)}$, that we may call the “Three Graces”, is the most interesting. Let us therefore focus on this subclass for the present illustration, and to simplify, let us choose monomials $f_2 = k_2 X^\alpha$, $f_4 = k_4 X^\beta$ and $s_4 = \kappa_4 X^\gamma$, where k_2 , k_4 and κ_4 are $\mathcal{O}(1)$ dimensionless constants, whose signs are imposed by the two cosmological equations (15). As underlined above, we must choose $\alpha \neq -1$ otherwise the $\mathcal{L}_{(2,0)}$ Lagrangian, Eq. (4a), is another bare cosmological constant. We see that the s_4 function behaves exactly as f_4 if $\gamma = \beta + 1$ and $\kappa_4 = k_4/2$. If both f_4 and s_4

are assumed to contribute with the same order of magnitude in these equations,³ then we need $\gamma = \beta + 1$, and everything behaves as if there were only f_2 and f_4 with k_4 replaced by $k_4 + 2\kappa_4$. We may thus consider only the case $\mathcal{L}_{(2,0)} + \mathcal{L}_{(4,0)}$. Then Eqs. (15) imply

$$M \propto \left[H^{-(\alpha+1)} (M_{\text{Pl}}^2 \Lambda_{\text{bare}})^{(\alpha-\beta-1)/2} \right]^{1/(\alpha-2\beta-3)}, \quad (18a)$$

$$X \propto \left[H^4 / (M_{\text{Pl}}^2 \Lambda_{\text{bare}}) \right]^{1/(\alpha-2\beta-3)}, \quad (18b)$$

$$|\dot{\varphi}| \propto \left[H^{1-\alpha} (M_{\text{Pl}}^2 \Lambda_{\text{bare}})^{(\alpha-\beta-2)/2} \right]^{1/(\alpha-2\beta-3)}, \quad (18c)$$

with $\mathcal{O}(1)$ numerical factors depending on the constants k_2 and k_4 and the exponents α and β . In the realistic case where $H \ll (M_{\text{Pl}}^2 \Lambda_{\text{bare}})^{1/4}$, we deduce thus that M , X and $\dot{\varphi}$ may be independently small or large depending on the positive or negative signs of the exponents of H in Eqs. (18). We find that M is small (with respect to M_{Pl}) if $\alpha < -1$ and $\beta < (\alpha - 3)/2$, or if $\alpha > -1$ and $\beta > (\alpha - 3)/2$. On the other hand, X is small (with respect to 1) if $\beta < (\alpha - 3)/3$. Finally, $|\dot{\varphi}|$ is small (with respect to M_{Pl}) if $\alpha < 1$ and $\beta < (\alpha - 3)/2$, or if $\alpha > 1$ and $\beta > (\alpha - 3)/2$. We quote below some even more specific examples to illustrate that these quantities can be independently large or small. Note that M must be either large or small, but never of the order of magnitude of $(M_{\text{Pl}}^2 \Lambda_{\text{bare}})^{1/4}$, because $\alpha = -1$ is forbidden. As underlined above, this behavior is actually valid for all combinations of Lagrangians (4) [unless we are in a limiting case which cannot predict the value of X]. Similarly, X may be either large or small, but never order 1 in the present $\mathcal{L}_{(2,0)} + \mathcal{L}_{(4,0)}$ model, otherwise this would correspond to some infinite exponent α or β . However, some other combinations of Lagrangians do allow for $X = \mathcal{O}(1)$, notably the $\mathcal{L}_{(4,0)} + \mathcal{L}_{(4,1)}$ and $\mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)}$ cases (without any $\mathcal{L}_{(2,0)}$), where $X = \mathcal{O}(1)$ is actually implied by the field equations. [These $\mathcal{L}_{(4,0)} + \mathcal{L}_{(4,1)}$ and $\mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)}$ combinations are also the only ones for which it is impossible to choose a small mass scale M , as one finds $M \propto H^{-1}$ in the first case and $M \propto H^{-3}$ in the second.] Finally, note that $|\dot{\varphi}| \sim (M_{\text{Pl}}^2 \Lambda_{\text{bare}})^{1/4}$ is possible in Eq. (18c), if one chooses $\alpha = 1$, i.e., $f_2(X) = k_2 X$.

Let us quote some more specific examples to illustrate their $\mathcal{O}(1)$ numerical factors and the relative sizes of their predictions.

For $f_2 = -1$ and $f_4 = 1$, we get (while neglecting $3H^2 = \Lambda_{\text{eff}}$ with respect to Λ_{bare})

$$M = (8H^2 M_{\text{Pl}}^2 \Lambda_{\text{bare}})^{1/6} \quad \Leftrightarrow \quad \Lambda_{\text{eff}} = \frac{3M^6}{8M_{\text{Pl}}^2 \Lambda_{\text{bare}}}, \quad (19a)$$

$$X = \frac{1}{6} \left(\frac{M_{\text{Pl}}^2 \Lambda_{\text{bare}}}{H^4} \right)^{1/3}, \quad (19b)$$

$$|\dot{\varphi}| = \frac{1}{\sqrt{6}} \left(\frac{M_{\text{Pl}}^2 \Lambda_{\text{bare}}}{H} \right)^{1/3}. \quad (19c)$$

Therefore, M is small but both X and $|\dot{\varphi}|$ are large in this model.

³ The Horndeski combination corresponds to $F_4 = 0$ in Eq. (6d), i.e., to $f_4 = 4s'_4$, therefore to $f_4 = k_4 X^\beta$ and $s_4 = \frac{1}{4} k_4 X^{\beta+1}/(\beta+1)$ in the present monomial case. For the cosmological background, it behaves thus as if there were only f_2 and f_4 with k_4 replaced by $(2\beta+3)k_4/(2\beta+2)$.

For $f_2 = -X^2$ and $f_4 = 1$, we get

$$M = \frac{4\sqrt{10} H^3}{M_{\text{Pl}}\sqrt{\Lambda_{\text{bare}}}} \Leftrightarrow \Lambda_{\text{eff}} = \frac{3}{2} \left(\frac{M^2 M_{\text{Pl}}^2 \Lambda_{\text{bare}}}{20} \right)^{1/3}, \quad (20a)$$

$$X = \frac{M_{\text{Pl}}^2 \Lambda_{\text{bare}}}{40 H^4}, \quad (20b)$$

$$|\dot{\varphi}| = 2H. \quad (20c)$$

Therefore, X is large but both M and $|\dot{\varphi}|$ are small in this model.

We shall see in Sec. VB that the models with $f_2 = k_2 X^\alpha$ and $f_4 = k_4 X^{-5/2}$ (or/and $s_4 = \kappa_4 X^{-3/2}$) have the quite interesting property that the observed Newton's constant G is not renormalized — i.e., that it is equal to the bare one entering action (3). Let us just quote here one example among them, say the nicely symmetric $f_2 = -X^{-3/2}$ and $s_4 = X^{-3/2}$. We get

$$M = 2\sqrt{3} H \Leftrightarrow \Lambda_{\text{eff}} = M^2/4, \quad (21a)$$

$$X = \left(\frac{24 H^4}{M_{\text{Pl}}^2 \Lambda_{\text{bare}}} \right)^2, \quad (21b)$$

$$|\dot{\varphi}| = \frac{48\sqrt{3} H^5}{M_{\text{Pl}}^2 \Lambda_{\text{bare}}}. \quad (21c)$$

Therefore, M , X and $|\dot{\varphi}|$ are all small in this model. Note that obtaining all M , X and $|\dot{\varphi}|$ small and keeping an unrenormalized Newton's constant are fully independent properties. For instance, the model $f_2 = 1$ and $f_4 = X^{-5/2}$ does also predict an unrenormalized G but gives $M \propto H^{-1/2}$ large while $X \propto H^2$ and $|\dot{\varphi}| \propto H^{1/2}$ are small. Note finally that negative powers are not problematic in our present cosmological context, since the background value of $\dot{\varphi}^2 = M^2 X$ does not vanish, and that X should thus not pass through zero. Negative powers are obviously a much more serious issue when considering Horndeski theories around a vanishing scalar background, and some surprising results of the literature [34] are actually related to such negative powers. Perturbations of the scalar field are very probably ill-defined in such cases.

The above examples show that the magnitude of $|\dot{\varphi}|$ depends crucially on the considered model. However, let us underline that the energy-momentum tensor $T^{\mu\nu}$ of this scalar field is *always* of order $M_{\text{Pl}}^2 \Lambda_{\text{bare}} g^{\mu\nu}$. Indeed, self-tuning means *by construction* that this energy-momentum tensor must almost compensate the bare cosmological constant in the Einstein equations (10). Therefore, whatever the values of M or $\dot{\varphi}$, we have anyway large $\mathcal{O}(M_{\text{Pl}})$ scalar effects, at least in the background.

We end this Section by mentioning another class of particular cases: models predicting an observed $3H^2 = \Lambda_{\text{eff}}$ which is fully independent from Λ_{bare} . In the $\mathcal{L}_{(2,0)} + \mathcal{L}_{(4,0)}$ subclass of models, Eq. (18a) shows that this happens when $\alpha = \beta + 1$, i.e., when $f_2 = k_2 X^\alpha$ and $f_4 = k_4 X^{\alpha-1}$. Indeed, one then gets $\Lambda_{\text{eff}} \approx M^2$, up to an $\mathcal{O}(1)$ numerical coefficient depending on the dimensionless constants k_2 , k_4 and α . Therefore, for a given theory with fixed M , the observed cosmological constant Λ_{eff} will remain unchanged even after a phase transition modifying Λ_{bare} . Note however that this necessarily means that the observed Hubble scale H is actually introduced by hand in the action, via the mass scale $M \approx H$, since H is independent of Λ_{bare} . Some fine-tuning is thus still required in this subclass of models, although it is now for a mass scale entering the generalized Horndeski Lagrangians,

and no longer for a vacuum energy whose quantum prediction cannot have the observed order of magnitude.

As underlined above Eq. (18), $\mathcal{L}_{(4,0)}$ and $\mathcal{L}_{(4,1)}$ give almost identical field equations. Predicting an observed Λ_{eff} independent from Λ_{bare} is thus obviously possible too in the $\mathcal{L}_{(2,0)} + \mathcal{L}_{(4,1)}$ subclass of models with $\alpha = \gamma$, i.e., with $f_2 = k_2 X^\alpha$ and $s_4 = \kappa_4 X^\alpha$. The particular case $\alpha = 0$, corresponding to $f_2 = -1$ and $s_4 = 1$, was actually the first model found with this property, in Ref. [24]. Another example of this kind is given in Eqs. (21) above, where $\Lambda_{\text{eff}} = M^2/4$ is indeed independent from Λ_{bare} . This behavior can also be obtained in most other combinations of Lagrangians $\mathcal{L}_{(n,p)}$, for instance with $f_2 = k_2 X^\alpha$ and either $f_3 = k_3 X^{\alpha-1/2}$, or $f_5 = k_5 X^{\alpha-3/2}$, or $s_5 = \kappa_5 X^{\alpha-1/2}$ (all other functions being assumed to vanish). The only particular cases for which it is not possible to predict Λ_{eff} independent from Λ_{bare} are again the $\mathcal{L}_{(4,0)} + \mathcal{L}_{(4,1)}$ and $\mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)}$ combinations, that we already mentioned in the paragraph below Eqs. (18): One always predicts $\Lambda_{\text{eff}} \propto \Lambda_{\text{bare}}$ in the first case, while $\Lambda_{\text{eff}} \propto \Lambda_{\text{bare}}^{2/3}$ in the second.

IV. SCHWARZSCHILD-DE SITTER SOLUTIONS

A. Self-tuning solutions around a spherical body

We now consider the same models as above, with the same cosmological behavior at large distances, but we study their predictions in the vicinity of a spherical and static massive body. Are they consistent with the Schwarzschild metric, which is very well tested at the first post-Newtonian order in the solar system? We do know that these models generically exhibit a Vainshtein mechanism, which reduces the observable scalar-field effects at small enough distances. But in the present self-tuning context, we saw in Sec. III that some quantities (like $\dot{\varphi}$) can take extremely large values, therefore the backreaction of the scalar field can *a priori* fully change the behavior of the metric, and solar-system tests are not guaranteed to be passed. Actually, one might even fear that none of these models is consistent with local tests, in spite of the Vainshtein mechanism.

A large number of works has been devoted in the literature to the Vainshtein mechanism in Galileon theories. However, most of the studies assumed a time-independent scalar field, see e.g. [35–40] (for a recent review on the Vainshtein mechanism see [23]). The Vainshtein mechanism with a time-dependent scalar has been considered in [22, 41], while Ref. [42] studied it in a subclass of beyond Horndeski theories (with $\mathcal{L}_{(5,0)} = \mathcal{L}_{(5,1)} = \Lambda_{\text{bare}} = 0$).

Our approach is quite different in the present Section. We shall scan the whole class of generalized Horndeski theories to look for a subclass which (i) is able to screen a huge cosmological constant Λ_{bare} , and (ii) reproduces the *exact* Schwarzschild-de Sitter solution of GR with a small but non-vanishing observed cosmological constant Λ_{eff} .

We choose to work in Schwarzschild coordinates

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (22)$$

and we consider a scalar field of the form

$$\varphi = \dot{\varphi}_c t + \psi(r), \quad (23)$$

where $\dot{\varphi}_c$ is now assumed to be a constant [contrary to Eqs. (14) and (15) above, which were valid for any time dependence]. This ansatz (23) allows us to separate time and radial

variables in all field equations because of the shift-symmetry of the theory. Indeed, since an undifferentiated φ cannot appear in these field equations, time derivatives are transformed into the constant $\dot{\varphi}_c$ (or 0), and we thus get only *ordinary* differential equations with respect to the radial coordinate. We also define the dimensionless ratio

$$q \equiv \frac{\dot{\varphi}_c}{M}, \quad (24)$$

and the cosmological value of the standard kinetic term (2) reads thus $X_c = q^2 = \text{const.}$

In the previous section, we saw that the 00-component of the Einstein equations and the scalar equation were enough to solve all field equations in FLRW. In the present static and spherically symmetric situation, three equations become necessary and sufficient. Indeed, the covariant conservation of the Einstein tensor $\nabla_\mu G^{\mu\nu} = 0$ implies now $2e^\lambda G_{\theta\theta}/r^3 = \partial_r G_{rr} + (2/r - \lambda' + \nu'/2)G_{rr} + \nu' e^{\lambda-\nu} G_{00}/2$, where a prime denotes radial differentiation, therefore the angle-angle components $G_{\phi\phi} = \sin^2 \theta G_{\theta\theta}$ are automatically solved once the 00 and rr components are, while all off-diagonal components vanish identically. As before, when taking into account the energy-momentum of the scalar field, this remains valid for $G^{\mu\nu} - T^{\mu\nu}/M_{\text{Pl}}^2$ instead of $G^{\mu\nu}$, up to terms proportional to the scalar equation $\nabla_\mu J^\mu = 0$, because of Eq. (11) implied by the diffeomorphism invariance of action (3).

We give in Appendix B the two relevant Einstein equations and the scalar current. Actually, we also simplified the rr -Einstein equation by combining it with the scalar current as in Eq. (10). In the present spherically symmetric case with $\dot{\varphi} = \text{const.}$, the scalar equation $\nabla_\mu J^\mu = 0$ simply reads $\partial_r(r^2 J^r) = 0$. Its solution is thus in general $J^r = C_r/r^2$, where C_r is an integration constant. However, since we assume that there is no bare matter-scalar coupling in action (3), the scalar field does not have any source term even within matter, therefore this integration constant must vanish (otherwise the scalar field would be singular at $r = 0$).

Since Eqs. (B1)–(B3) are quite heavy, we checked them again by two independent methods: First by deriving the full covariant equations and specifying them to metric (22); and second the “minisuperspace” technique, in which the form (22) is imposed directly within the action, and then one varies it with respect to the three fields $\nu(r)$, $\lambda(r)$ and $\varphi'(r) = \psi'(r)$.

Our aim is now to exhibit a subclass of models which is consistent with an exact Schwarzschild-de Sitter metric. We therefore impose so in Eqs. (B1)–(B3), by enforcing the metric to take the form (22) with

$$e^\nu = e^{-\lambda} = 1 - \frac{r_s}{r} - (Hr)^2, \quad (25)$$

where H is the Hubble rate (assumed to be constant), related to the observed cosmological constant by $\Lambda_{\text{eff}} = 3H^2$. These field equations (B1)–(B3) then become long expressions depending on the radial coordinate r and radial derivatives of the scalar field (23). However, we noticed that an extra hypothesis simplifies them tremendously. In addition to the above assumptions (22), (23) and (25), we will also restrict to the case where $X \equiv -(\varphi_\lambda)^2/M^2$ remains constant everywhere, even in the vicinity of the massive body. This means that we simply impose

$$X = X_c, \quad (26)$$

where $X_c = q^2$ is the constant cosmological value of X . All functions of X entering Eqs. (B1)–(B3) then obviously become constants, whose precise values are still unknown, but which do not depend any longer on the radial coordinate r . Moreover, since we have

$X = e^{-\nu} q^2 - e^{-\lambda} \varphi'^2 / M^2$ in Schwarzschild coordinates (22), where $\varphi' \equiv \partial_r \varphi = \psi'$ denotes the radial derivative of the scalar field (23), we may also replace any occurrence of φ' by the square root⁴ of

$$\varphi'^2 = e^\lambda (e^{-\nu} - 1) M^2 X_c, \quad (27)$$

which is a *known* function of r . Its radial derivative also gives us the exact expression of $\varphi'' \equiv \partial_r^2 \varphi$. Therefore, thanks to the greatly simplifying hypothesis (26), the field equations (B1)–(B3) now become mere functions of r alone, involving some unknown constants depending of the functions $f_n(X)$, $s_n(X)$ and their derivatives (with respect to X , but also evaluated at $X = X_c$). Since these field equations must be satisfied at any spacetime point, it is then straightforward to extract from them some necessary conditions on the functions $f_n(X)$ and $s_n(X)$. For instance, an expansion of Eqs. (B1)–(B3) in powers of $(r - r_0)$ around any radius r_0 (even $r_0 = 0$) suffices to prove that some combinations of $f_n(X)$, $s_n(X)$ and their derivatives *must* vanish. After having derived such necessary conditions, one may plug them back into Eqs. (B1)–(B3) to check whether they also suffice. If the field equations do not vanish identically, this means that other conditions still need to be imposed. This procedure allowed us to prove that the following conditions are necessary and sufficient for the ansatz (22), (23), (25) and (26) to be consistent with all field equations (B1)–(B3):

$$-Xf_2 + 6 \left(\frac{H}{M} \right)^2 [X^2 f_4 + 2Xs_4] = \frac{M_{\text{Pl}}^2}{M^4} (\Lambda_{\text{bare}} - 3H^2), \quad (28a)$$

$$[Xf_2]' + 6 \left(\frac{H}{M} \right)^2 [X^2 f_4 + 2Xs_4]' = 0, \quad (28b)$$

$$Xf_5 + 2s_5 = 0 \quad \text{and} \quad [Xf_5 + 2s_5]' = 0, \quad (28c)$$

$$[X^{3/2} f_3]' = 0, \quad (28d)$$

where as before all functions f_n and s_n depend on X , and a prime denotes differentiation with respect to X . This is therefore a particular case of Eqs. (15) that we obtained in cosmology, which is not a surprise since the asymptotic behavior of the present solution at large radii should match with this cosmological solution. But we find here some restrictions with respect to (15): The f_3 function needs to be very precisely tuned at the cosmological value $X = X_c$ (in order not to contribute to any background equation at this precise value), while f_5 and s_5 should be related in a specific way at this value of X (again so that their sum $Xf_5 + 2s_5$ does not contribute to any background equation).

It should be underlined that this set of equations (28) only needs to be satisfied *at* the cosmological value $X = X_c$, and notably that (28c) and (28d) should *not* be imposed for all X . Actually, if Eq. (28d) were satisfied for all X , then the Lagrangian $\mathcal{L}_{(3,0)}$, Eq. (4b), would be a total derivative, as underlined in Sec. III above, and it would not contribute to any observable. On the other hand, the sum $\mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)}$, Eqs. (4d) and (4f), would not be a total derivative even if the two conditions (28c) were imposed for all X . It just happens that this combination does not contribute to the field equations when imposing both spherical symmetry and $X = \text{const.}$, as in the present section. Note that this combination, satisfying conditions (28c) for all X , is not the Horndeski one either, which would correspond

⁴ Obviously, the r.h.s. of Eq. (27) needs to be positive for such an equation to make sense, otherwise this would correspond to unstable configurations. For example, such a situation takes place in the model considered in [43] when the bare cosmological constant is absent.

to $s'_5(X) = \frac{3}{4}f_5(X)$ (unless we are in the limiting case $f_5 \propto X^{-5/2}$ and $s_5 \propto X^{-3/2}$). The conditions (28c) and (28d), at $X = X_c$, mean thus that f_3 , f_5 and s_5 do not contribute to our Schwarzschild-de Sitter background solution, but they do change the behavior of perturbations around this background, and they also change the dynamics before the background reaches its equilibrium configuration.

Equation (28a) may also be rewritten as the expression of the effective (observed) cosmological constant $\Lambda_{\text{eff}} = 3H^2$ in terms of $X = X_c$ and the bare cosmological constant:

$$\Lambda_{\text{eff}} = \frac{\Lambda_{\text{bare}} + \frac{M^4}{M_{\text{Pl}}^2} X f_2}{1 + 2 \left(\frac{M}{M_{\text{Pl}}} \right)^2 (X^2 f_4 + 2X s_4)}. \quad (29)$$

This form underlines that $X f_2$ acts as an additive constant to Λ_{bare} (recall that if $f_2 \propto 1/X$, then the Lagrangian $\mathcal{L}_{(2,0)}$ would be another trivial bare cosmological constant), whereas $X^2 f_4$ and $X s_4$ can be understood as renormalization factors. However, Eq. (29) cannot be interpreted so directly because these functions are anyway related via Eq. (28b), and this explains notably why we found models in which Λ_{eff} is fully independent from Λ_{bare} , at the end of Sec. III. In the realistic case where $3H^2 = \Lambda_{\text{eff}} \ll \Lambda_{\text{bare}}$, one may of course neglect $3H^2$ in the r.h.s. of Eq. (28a), and the added 1 in the denominator of Eq. (29) may thus be suppressed.

The conclusion of the present subsection is that a subclass of beyond Horndeski theories does provide both cosmological self-tuning, and a local metric around a spherical body which is indistinguishable from GR plus a small Λ_{eff} . This subclass depends on all six functions f_n and s_n defining beyond Horndeski theories, Eqs. (4), but they should satisfy the five relations (28) at the background value $X = X_c$. Three of them, f_2 , f_4 and s_4 , are responsible for the self-tuning, because Eqs. (28a) and (28b) generically fix both the values of X_c and $3H^2 = \Lambda_{\text{eff}}$ (see Sec. III for a discussion of the non-generic cases in which one of those is not predicted). We shall call them the “Three Graces”. The three other functions, f_3 , f_5 and s_5 , play a passive rôle both for the cosmological background and the local spherically symmetric solution, provided they satisfy Eqs. (28c) and (28d) at $X = X_c$. Note however that the latter three “stealth” Lagrangians do contribute to the dynamics of perturbations around our exact solutions, and also to the time evolution of the Universe before the equilibrium value $X = X_c$ is reached.

B. Black hole solutions

An interesting byproduct of the above Sec. IV A is the existence of regular black hole solutions. Indeed, when conditions (28) are imposed at $X = X_c$, then the field equations admit the exact Schwarzschild-de Sitter solution (22) and (25) — with an observed Λ_{eff} much smaller than Λ_{bare} . If no matter source is assumed for $r > r_s$, the metric has thus the same form as that of a general relativistic black hole.

Let us summarize here in which conditions such black holes exist. We consider the most general beyond Horndeski Lagrangian

$$\mathcal{L}_{(2,0)} + \mathcal{L}_{(3,0)} + \mathcal{L}_{(4,0)} + \mathcal{L}_{(4,1)} + \mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)}, \quad (30)$$

defined by Eqs. (1) and (4), but we require the following three conditions at the cosmologi-

cally imposed value $X = X_c = q^2$ of $X \equiv -(\varphi_\lambda)^2/M^2$:

$$\begin{aligned} X f_5 + 2s_5 &= 0, \\ [X f_5 + 2s_5]' &= 0, \\ [X^{3/2} f_3]' &= 0. \end{aligned} \tag{31}$$

On the other hand, the three other functions f_2 , f_4 and s_4 are free, and they fix the value of X_c from Eqs. (28a) and (28b). Note that conditions (31) do not need to be satisfied for all values of X , but only at $X = X_c$. The metric (22) reads then

$$e^\nu = e^{-\lambda} = 1 - \frac{r_s}{r} - (Hr)^2, \tag{32}$$

with $\Lambda_{\text{eff}} = 3H^2$ given by Eq. (29). The solution for the scalar field is such that

$$X = q^2 = \text{const.}, \tag{33}$$

so that using the ansatz (23), one finds explicitly

$$\varphi = qMt + \psi(r), \tag{34}$$

with

$$\psi' = \pm \frac{qM \sqrt{r_s/r + (Hr)^2}}{1 - r_s/r - (Hr)^2}. \tag{35}$$

Although this last exact expression may be explicitly integrated, let us quote here only the solution for a negligible value of H :

$$\varphi = qM \left\{ t \pm \left[2\sqrt{r_s r} + r_s \ln \left(\frac{\sqrt{r} - \sqrt{r_s}}{\sqrt{r} + \sqrt{r_s}} \right) \right] + \mathcal{O}(H^2) \right\}. \tag{36}$$

The regularity of such black-hole solutions is easy to understand. First of all, since the metric is of the Schwarzschild-de Sitter form, it is clear that the backreaction of the scalar field on the metric is everywhere finite, including both on the event and the cosmological horizons. In fact, for these solutions, the energy-momentum tensor of the scalar field (8a) takes precisely the form of a vacuum energy,

$$\frac{T_{\mu\nu}}{M_{\text{Pl}}^2} = (\Lambda_{\text{bare}} - \Lambda_{\text{eff}}) g_{\mu\nu}, \tag{37}$$

as is obvious from our exact Schwarzschild-de Sitter solution for the metric. Concerning the regularity of the scalar field itself, let us underline that the invariants X and $J_\mu J^\mu$, involving the derivatives of φ , are regular everywhere. Indeed $X = q^2 = \text{const.}$ by construction, while the current J^μ actually fully vanishes for the present black hole solutions. The reason is that by construction⁵ $J^r = 0$, while J^0 can be checked to be also proportional to Eq. (28b) in the present case, therefore the invariant $J^2 = J_\mu J^\mu$ vanishes everywhere. The regularity of this norm of the current is an additional condition which becomes important notably if

⁵ For a static and spherically symmetric black hole with the time-dependence ansatz (34), Ref. [44] proved that $J^r = 0$ follows from the 0r-Einstein equation.

matter is assumed to be directly coupled to J^μ . It is also one of the key assumptions for the no-hair theorem for Galileons when the scalar field is time-independent [45] (contrary to our present framework).

The black hole solutions presented here can be considered as generalizations of the self-tuning solutions first found for the Lagrangian containing the “John” term $\mathcal{L}_{(4,1)}$ [24]. More specifically, this reference studied the model $\mathcal{L}_{(2,0)} + \mathcal{L}_{(4,1)} + \Lambda_{\text{bare}}$, with $f_2(X) = \text{const.}$ and $s_4(X) = \text{const.}$, and showed that it admits an exact Schwarzschild-de Sitter solution describing a regular black hole. Generalizations of this self-tuning solution have been found later: Ref. [16] concluded that an arbitrary $s_4(X)$ gives a similar solution, while the authors of Ref. [28] considered the Horndeski combination of $\mathcal{L}_{(4,0)}$ and $\mathcal{L}_{(4,1)}$, i.e., with $s'_4 = \frac{1}{4}f_4$ (see the recent review [46]).

As a final remark, let us comment on our results above when the $\mathcal{L}_{(3,0)}$ term is present. Equations (31) show that an exact Schwarzschild-de Sitter solution exists only for a particular class of functions f_3 . This explains why Ref. [47] could not find simple solutions for the theory with the simplest cubic Galileon, i.e., $f_3 = 1$. In this case $f_3 = 1$, conditions (31) imply $q = X_c^{1/2} = 0$, which is inconsistent with our hypothesis of a time-dependent scalar field (34).

V. PERTURBATIONS

A. Backreaction of the scalar field

The aim of the present Section is to go beyond the exact solutions of the previous one, i.e., the Schwarzschild-de Sitter metric predicted in the “Three Graces” of Eqs. (28) when assuming $X = \text{const.}$ Can other models predict a metric which is close enough to the Schwarzschild solution at small radii, so that solar-system tests are passed? An obvious first answer is to add small corrective terms to the Three Graces, i.e., to assume that beyond the three functions f_2 , f_4 and s_4 satisfying *approximately* Eqs. (28a) and (28b), one adds *small enough* extra functions f_3 , f_5 and s_5 which do not respect conditions (28c) and (28d).

But is it possible to pass solar-system tests in models which differ significantly from the Three Graces (28)? To answer this question, we shall adopt here a perturbative approach, whose spirit may be summarized as follows. In order to pass solar-system tests, the metric should be close to the Schwarzschild solution at small radii. We may thus assume that it takes this approximate form to solve the scalar equation, and then plug this scalar solution into the Einstein equations to estimate its backreaction on the metric. Any contradiction will prove that our approximations are not valid, i.e., that solar-system tests cannot be passed.

We start from the most general field equations in spherical symmetry, given in Eqs. (B1)–(B3) of Appendix B. Instead of expanding them around the Schwarzschild solution, it is actually more convenient to assume that the metric is almost flat, i.e., that the functions λ and ν entering (22) and their radial derivatives are small with respect to 1. Our approximate Einstein equations should therefore reproduce ultimately the linearized behavior of the Schwarzschild solution. A mere linearization would not be consistent for the scalar field itself, on the other hand. Indeed, we know that nonlinear effects are crucial in Galileon and Horndeski theories, for which a Vainshtein mechanism generically exists at small radii. Our approximation scheme should therefore take into account the powers of the radial derivative φ' entering the field equations. The only hypothesis that we shall make is $|\varphi'| \ll |\dot{\varphi}| = qM$,

but we *a priori* do not know the order of magnitude of the second derivative φ'' , and φ'^2 or higher powers of φ' are *not* assumed to be negligible with respect to λ nor ν . Finally, since the various Lagrangians $\mathcal{L}_{(n,p)}$ involve different functions of X , whose magnitude can be very different, we treat each of them separately, without comparing the respective terms they generate. But since we assume that none of the functions $f_n(X)$ nor $s_n(X)$ involve large dimensionless parameters, we can consider that $Xf'_n \sim f_n$ for each of them separately.

It should be underlined that our hypothesis $|\varphi'| \ll |\dot{\varphi}|$ might be problematic in the models where $\dot{\varphi} = qM$ is predicted to be very small, since local perturbations by the massive body might happen to be larger than such a small cosmological background. Models involving negative powers of X might even yield to singularities in this case, since X may pass through zero, between its positive value at cosmologically large distances and a negative one in the vicinity of the massive body. In the following, we should thus trust our perturbative treatment mainly when the cosmologically predicted $|\dot{\varphi}|$ is not too small. However, we will see below that in the most interesting subclass of models (the Three Graces), the approximation $|\varphi'| \ll |\dot{\varphi}|$ is actually justified even when $|\dot{\varphi}|$ is predicted to be extremely small (with respect to the Planck mass), and even if the Lagrangian involves negative powers of X . Our perturbative treatment has thus clearly a wider range of application than naively expected. In any case, one should keep in mind that the conclusions of the present Section are valid only if our hypothesis $|\varphi'| \ll |\dot{\varphi}|$ is satisfied.

The approximation scheme described above (linearization around a flat metric, and for each different form of term, keeping the lowest order in powers of $\varphi'/\dot{\varphi}$) then transforms Eqs. (B1)–(B3) into the following form. It happens that the time-time component of the Einstein equation (9a) can be easily integrated once with respect to the radial coordinate, at this approximation. We thus quote below its radial integral, multiplied by a global factor $M_{\text{Pl}}^2 r^2$ to simplify its expression. We then give the radial-radial component of the linear combination (10) multiplied by a factor $M^4 M_{\text{Pl}}^2 r^2$, and finally the radial component of the scalar current (8b) multiplied by a factor $-2M^2 r^2$:

$$\begin{aligned} M_{\text{Pl}}^2 r \left(\lambda - \frac{1}{3} \Lambda_{\text{bare}} r^2 \right) + \frac{1}{3} M^4 X r^3 (f_2 + 2X f'_2) + M^2 X r^2 (3f_3 + 2X f'_3) \varphi' \\ + 2X r (5f_4 + 2X f'_4) \varphi'^2 + 2X (7f_5 + 2X f'_5) \varphi'^3 / M^2 \\ + 4M^2 X r \lambda (s_4 + 2X s'_4) + 8r s_4 \varphi'^2 + 4X \lambda (3s_5 + 2X s'_5) \varphi' + 8s_5 \varphi'^3 / M^2 = \frac{m}{4\pi}, \end{aligned} \quad (38a)$$

$$\begin{aligned} M^4 M_{\text{Pl}}^2 (-\lambda + r\nu' + \Lambda_{\text{bare}} r^2) + M^8 X r^2 f_2 \\ - 2M^4 X (\varphi' + 2r\varphi'') \varphi' f_4 + 6 (\nu' \varphi'^3 - 2M^2 X \varphi'') \varphi'^2 f_5 \\ + 4M^4 [M^2 X \lambda s_4 + M^2 X r (s_4 + 2X s'_4) \nu' - 2s_4 \varphi'^2 + 4X r s'_4 \varphi' \varphi''] \\ + 4M^2 X [M^2 (3s_5 + 2X s'_5) \nu' + 4s'_5 \varphi' \varphi''] \varphi' = 0, \end{aligned} \quad (38b)$$

$$\begin{aligned} 4M^4 r^2 (f_2 + X f'_2) \varphi' - M^2 r (3f_3 + 2X f'_3) (M^2 X r \nu' - 4\varphi'^2) \\ - 4M^2 X r [f_4 \lambda' + (5f_4 + 2X f'_4) \nu'] \varphi' + 8(2f_4 + X f'_4) \varphi'^3 \\ - 6X [2f_5 \lambda' + (7f_5 + 2X f'_5) \nu'] \varphi'^2 + 16M^2 [X (\lambda + r\lambda') s'_4 + (\lambda - r\nu') s_4] \varphi' \\ - 4M^2 X \lambda (3s_5 + 2X s'_5) \nu' + 8(2X s'_5 \lambda' - 3s_5 \nu') \varphi'^2 = 0. \end{aligned} \quad (38c)$$

Here X denotes the cosmological background $X_c = q^2$, although we did not write its index to simplify the notation, and all functions f_n and s_n , as well as their derivatives, are evaluated

at X_c . Beware that the primes denote derivatives with respect to the argument of the corresponding terms, i.e., $f'_n = df_n(X)/dX$ but $\varphi' = \partial_r \varphi$, $\varphi'' = \partial_r^2 \varphi$, $\lambda' = \partial_r \lambda$ and $\nu' = \partial_r \nu$.

For the same reason as in Sec. IV above, i.e., because we assume there does not exist any direct matter-scalar coupling in action (3), we know that $J^r = 0$ in the present static and spherically symmetric situation, and this explains why Eqs. (38b) and (38c) have vanishing right-hand sides. On the other hand, the right-hand side $m/(4\pi)$ of Eq. (38a) is imposed by the matching of this equation with the interior of the massive body, whose total mass is denoted m . The matter contribution to T_{00} is indeed the matter density ρ (at this order of approximation), and we have $\int \rho r^2 dr = m/(4\pi)$.

The analysis of Eqs. (38) can be decomposed in three different cases, depending on which beyond Horndeski Lagrangians dominate at small distances. It is indeed expected that only one of them dominates locally [although it may happen that several of them simultaneously dominate, when their functions $f_n(X)$ or $s_n(X)$ are tuned to obtain such a behavior]. For instance, the local domination of the $\mathcal{L}_{(3,0)}$ term would be related to the well-known Vainshtein mechanism. It should thus be kept in mind that the cosmological background (and notably the predicted value of Λ_{eff}) may not depend on the same set of terms as those which dominate at small distances. We will thus in general treat the local equations (38) without assuming that the same functions are responsible for the cosmological background.

The first case one may consider is when f_2 , f_4 and/or s_4 dominate at small distances. This corresponds to the Three Graces, and our results of Sec. IV show that an exact Schwarzschild-de Sitter solution is then possible. It is thus obvious that the linearized equations (38) are also consistent with a local Schwarzschild metric, and it is not necessary to check again so. We will see in Sec. VB below that these linearized equations (38) are nevertheless useful in this Three Graces case too, to study the renormalization of Newton's constant.

The second case we consider is when f_3 happens to dominate at small distances. Then Eq. (38c) tells us that either $3f_3 + 2Xf'_3 = 0$ or $4\varphi'^2 = M^2 X r \nu'$. But if $3f_3 + 2Xf'_3 = 0$, then f_3 is fully passive (cf. our cosmological discussion in Sec. III, and the fact that $\mathcal{L}_{(3,0)}$ is a total derivative when such a condition is imposed for all X), and it cannot dominate at small distances. Therefore, we must have $4\varphi'^2 = \dot{\varphi}^2 r \nu'$, and if we assume that the metric is approximately of the Schwarzschild form (to pass solar-system tests), i.e., $\nu \approx -r_s/r$, we thus get $4\varphi'^2 = \dot{\varphi}^2 r_s/r$. Plugging this back into Eq. (38a), we find that the backreaction of the scalar is

$$M^2 X r^2 (3f_3 + 2Xf'_3) \varphi' = \frac{1}{2} \dot{\varphi}^3 (3f_3 + 2Xf'_3) \sqrt{r_s r^3}. \quad (39)$$

This is to be compared to the r.h.s. of Eq. (38a), namely $m/(4\pi)$. Depending on which Lagrangians determine the cosmological evolution, it may happen that this backreaction is negligible, and therefore that solar-system tests can be passed [although this situation would need a well-chosen function $f_3(X)$]. However, this is not the case when $\mathcal{L}_{(3,0)}$ also contributes significantly to cosmology. Let us illustrate so on the simple example of $\mathcal{L}_{(2,0)} + \mathcal{L}_{(3,0)}$ with monomials $f_2 = k_2 X^\alpha$ and $f_3 = k_3 X^\beta$. Then the cosmological equations (15) imply that we *always* have $\dot{\varphi}^3 f_3 \sim M_{\text{Pl}}^2 \Lambda_{\text{bare}}/H$, up to $\mathcal{O}(1)$ factors, whatever the exponents α and β entering the monomials. Therefore, the backreaction (39) is always of order $M_{\text{Pl}}^2 \Lambda_{\text{bare}} \sqrt{r_s r^3}/H \sim [(\Lambda_{\text{bare}}/H^2)(\Lambda_{\text{bare}} r_s^2)(r/r_s)^3]^{1/2} m$, which is much larger than m because the term within the square brackets is a product of three large numbers.⁶ In conclusion, in

⁶ If $M_{\text{Pl}}^2 \Lambda_{\text{bare}}$ is assumed to take the smallest possible theoretical prediction, namely $|\rho_{\text{QCD}}| \sim 10^{-2} \text{GeV}^4$,

this simple $\mathcal{L}_{(2,0)} + \mathcal{L}_{(3,0)}$ model, the metric cannot be close to the Schwarzschild solution, and solar-system tests are not passed. The only ways out are either that the contribution of $\mathcal{L}_{(3,0)}$ is negligible in the cosmological equations (15), so that $\dot{\varphi}$ is actually unrelated to f_3 and the backreaction $\propto \dot{\varphi}^3 f_3$ can be small enough, or that other Lagrangians than $\mathcal{L}_{(3,0)}$ dominate at small distances, which depends on the functions $f_n(X)$ entering them.

The third and final case is when f_5 and/or s_5 dominate at small distances. If we assume an approximate Schwarzschild metric, then Eq. (38c) implies

$$\varphi'^2 = \frac{-2M^2 X r_s (3s_5 + 2X s'_5)}{r [3X (5f_5 + 2X f'_5) + 4(3s_5 + 2X s'_5)]}. \quad (40)$$

Note that f_5 alone (with $s_5 = 0$) is not allowed to dominate in the vicinity of the massive body, otherwise its contribution would violate Eq. (38c). [The only way out would be to impose $(X^{5/2} f_5)' = 0$, in which case it would actually not dominate locally.] In fact, s_5 alone (with $f_5 = 0$) is not allowed either to dominate locally, otherwise Eq. (40) would give a negative φ'^2 . We should thus assume that both f_5 and s_5 dominate simultaneously. Plugging the expression (40) of φ'^2 into Eq. (38a) gives us the backreaction of the scalar field on the metric

$$-\frac{8}{M^2} [X(2f_5 + X f'_5) + 2(s_5 + X s'_5)] \varphi'^3, \quad (41)$$

[with φ' still given by Eq. (40)], which is again to be compared to $m/(4\pi)$, i.e., the r.h.s. of Eq. (38a). Similarly to the case of $\mathcal{L}_{(3,0)}$ above, it may happen that this backreaction is negligible if the cosmology is determined by other Lagrangians than $\mathcal{L}_{(5,0)}$ or $\mathcal{L}_{(5,1)}$, although f_5 and s_5 are assumed to dominate at small distances [this would also need some well-chosen functions $f_5(X)$ and $s_5(X)$]. But if these Lagrangians do contribute significantly to the cosmological background, then we face again the same difficulty as for $\mathcal{L}_{(3,0)}$: The backreaction of the scalar field is much larger than the central source $m/(4\pi)$. This can be illustrated on the simple example of $\mathcal{L}_{(2,0)} + \mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)}$ with monomials $f_2 = k_2 X^\alpha$, $f_5 = k_5 X^\beta$ and $s_5 = \kappa_5 X^\gamma$. Then one finds that the backreaction (41) is always of order $M_{\text{Pl}}^2 \Lambda_{\text{bare}} H^{-3} (r_s/r)^{3/2} \sim [(\Lambda_{\text{bare}}/H^2)^3 (\Lambda_{\text{bare}} r_s^2) (Hr)^{-6}]^{1/4} m$, which is much larger than m because the term within the square brackets is a product of three large numbers.

However, even when such an *a priori* large backreaction is expected, there still exists one possibility to pass solar-system tests. It suffices that

$$X(2f_5 + X f'_5) + 2(s_5 + X s'_5) = 0, \quad (42)$$

since this factor multiplies the backreaction (41). On the other hand, note that it would not be possible to impose $(3s_5 + 2X s'_5) = 0$ [cf. our limiting case discussed in Sec. III below Eqs. (15)], although this would also give a vanishing backreaction. Indeed, this would correspond to $\varphi' = 0$ in Eq. (40), in contradiction with our hypothesis that f_5 and s_5 dominate the local physics of φ . But condition (42) may be imposed without any inconsistency nor obtaining a trivial model. One can also check that the dominant (f_5 and s_5) terms of the

then $\Lambda_{\text{bare}} r_s^2$ would actually be of order $\mathcal{O}(1)$ for the Schwarzschild radius of the Sun, but this is anyway multiplied by the large factors $\Lambda_{\text{bare}}/H^2$ and $(r/r_s)^3$. Let us also mention that Newton's constant G is not renormalized in the present model, contrary to those discussed in Sec. V B below, and therefore that it is legitimate to identify here r_s and $2Gm = m/(4\pi M_{\text{Pl}}^2)$.

second Einstein equation (38b) also vanish when this condition is assumed. The two Einstein equations therefore reduce to those of general relativity when condition (42) is imposed, and Schwarzschild solution is recovered at small distances.

Note that Eq. (42) would be a consequence of the two conditions (28c) we found to get our exact solution of Sec. IV, but it does not suffice to imply both of them. In the present approximation scheme, we find thus that less constraints are needed to predict a Schwarzschild solution. It is probable that a higher-order analysis, taking into account first post-Newtonian terms in the g_{00} component of the metric [which are of order $(r_s/r)^2$], would imply a second condition, and that we would then recover the two of Eqs. (28c). But at the present linear order in r_s , the only conclusion we can draw is that the combination of Lagrangians $\mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)}$ only needs to satisfy the single condition (42) to be consistent with a Schwarzschild metric when they dominate locally, whatever the cosmological behavior [which may depend on other Lagrangians $\mathcal{L}_{(n,p)}$] and even if it yields very large factors multiplying the f_5 and s_5 terms in the local equations.

In conclusion, when $\mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)}$ dominate the behavior of φ in the vicinity of a massive body, there are two ways to pass solar-system tests. The first one is similar to the case of $\mathcal{L}_{(3,0)}$ above, namely when the cosmological evolution, depending on other Lagrangians $\mathcal{L}_{(n,p)}$, is such that the backreaction (41) is small enough with respect to the mass m of the body (but this needs some well-chosen functions f_5 and s_5). The second possibility is to choose a model satisfying condition (42), which is a subset of Eqs. (28c) found for the exact solutions of Sec. IV. Then the scalar field does not backreact at all on the metric (when f_5 and s_5 locally dominate) whatever the cosmological solution.

B. Renormalization of Newton's constant

Although the quantity $2Gm = m/(4\pi M_{\text{Pl}}^2)$ entering Eq. (38a) would be called the Schwarzschild radius of the body in standard general relativity, one should keep in mind that in the present class of theories, this is *not* the coefficient entering the possible $\mathcal{O}(1/r)$ terms in $-g_{00}$ and g_{rr} . Indeed, the scalar field also contributes crucially to the behavior of the metric, and one does not even predict a Newtonian potential $\propto 1/r$ in most models. Even in the exact solutions of Sec. IV where the metric happens to take the Schwarzschild-de Sitter form, Eqs. (22) and (25), the Schwarzschild radius r_s entering its expression does generically differ from $2Gm$.

Let us indeed consider the particular case in which only f_2 , f_4 and s_4 dominate at small enough distances, i.e., the Three Graces of Eqs. (28). Let us also assume that $X = q^2 = \text{const.}$, like in Sec. IV, which implies

$$\varphi'^2 = e^\lambda(e^{-\nu} - 1)M^2q^2 = M^2q^2r_s/r + \mathcal{O}(r_s^2/r^2) + \mathcal{O}(\Lambda_{\text{eff}}r^2). \quad (43)$$

Note that this means we always have $\varphi'^2 \ll \dot{\varphi}^2 = M^2q^2$, i.e., the condition we assumed to make the expansions of Sec. V, even in the cases where $|\dot{\varphi}|$ will be predicted to be extremely small with respect to the Planck mass.

Then, the constant contributions to Eq. (38a) (neglecting those $\propto r^3$ which dominate at larger distances) imply

$$r_s = \frac{2Gm}{1 + 4 \left(\frac{M}{M_{\text{Pl}}} \right)^2 X^{1/2} [X^{5/2}f_4 + 2X^{3/2}s_4]'}, \quad (44)$$

where the prime denotes derivation with respect to X . This is equivalent to a renormalization of Newton's constant G by the denominator of (44). This renormalization does depend on the cosmological background via X , but note that it is body independent. In other words, it cannot be distinguished from general relativity by local experiments, even by equivalence principle tests involving three bodies or more. It suffices that the ratio of the bare gravitational constant G and the denominator of (44) take the experimental value of Newton's constant. [Note that we are talking here only of the non-observable effect caused by this renormalization of G . There may exist other deviations from GR in three-body systems, for instance preferred-frame effects, that we do not discuss in the present paper.]

In the realistic situation where the observed Λ_{eff} is much smaller than Λ_{bare} , the added 1 in the denominator of Eq. (44) is generically negligible. It is indeed dominated by the second term involving functions of X , which is of the same order of magnitude as those entering the cosmological equations (15), or more precisely Eqs. (28a) and (28b) in the present Three Graces. Combining these equations with (44), we thus generically predict

$$(M_{\text{Pl}}^{\text{bare}})^2 \Lambda_{\text{bare}} \sim (M_{\text{Pl}}^{\text{eff}})^2 \Lambda_{\text{eff}}, \quad (45)$$

up to $\mathcal{O}(1)$ numerical factors, where $M_{\text{Pl}}^{\text{bare}}$ means our previous notation M_{Pl} , while $M_{\text{Pl}}^{\text{eff}}$ is the numerical value corresponding to the actually measured Newton's constant. For instance, in example (19), one gets $5 (M_{\text{Pl}}^{\text{bare}})^2 \Lambda_{\text{bare}} = 3 (M_{\text{Pl}}^{\text{eff}})^2 \Lambda_{\text{eff}}$, while example (20) gives $3 (M_{\text{Pl}}^{\text{bare}})^2 \Lambda_{\text{bare}} = (M_{\text{Pl}}^{\text{eff}})^2 \Lambda_{\text{eff}}$. Let us recall that quantum field theory predicts the value of the *vacuum energy density* from the matter action S_{matter} of Eq. (3). Although we decide to write it as a product $(M_{\text{Pl}}^{\text{bare}})^2 \Lambda_{\text{bare}}$ in this action, it is *a priori* unrelated to Newton's constant nor to the observed accelerated expansion of the Universe. The cosmological constant problem is precisely that the measured values of G (e.g. by Cavendish experiments) and of the cosmological constant (e.g. from type-Ia supernovae data) give a product $(M_{\text{Pl}}^{\text{eff}})^2 \Lambda_{\text{eff}}$ much too small, by many orders of magnitude, with respect to the predicted vacuum energy density $(M_{\text{Pl}}^{\text{bare}})^2 \Lambda_{\text{bare}}$. In the present scenario, Eq. (45) implies thus that the cosmological constant problem is not solved at all, and not even alleviated: The observable quantity $(M_{\text{Pl}}^{\text{eff}})^2 \Lambda_{\text{eff}}$ actually keeps the same order of magnitude as the huge bare vacuum energy density!

However, the generic behavior (45) is no longer valid if the denominator of Eq. (44) is not large, and this can happen without any fine tuning if the functions f_4 and s_4 are chosen so that

$$[X^{5/2} f_4 + 2X^{3/2} s_4]' = 0, \quad (46)$$

at $X = X_c$. This condition obviously reduces the space of allowed models, but it does not need any large nor small dimensionless number to be imposed. The combination $X^{5/2} f_4 + 2X^{3/2} s_4$ itself must not vanish, otherwise the field equations (28a) and (28b) cannot be satisfied (unless $f_2 \propto 1/X$, meaning that $\mathcal{L}_{(2,0)}$, Eq. (4a), is a second bare cosmological constant). We must thus choose

$$X^{5/2} f_4 + 2X^{3/2} s_4 = \text{const.} \quad (47)$$

Many possibilities exist in which f_4 and s_4 almost compensate each other apart from this constant, but they all give the same physics both in the cosmological framework of Sec. IV and in our exact solutions for spherical symmetry of Sec. V. It suffices thus to consider the simplest cases of $f_4 = k_4 X^{-5/2}$ and/or $s_4 = \kappa_4 X^{-3/2}$, where k_4 and κ_4 are dimensionless

constants of order 1. Then, Eq. (44) implies that we have strictly $M_{\text{Pl}}^{\text{bare}} = M_{\text{Pl}}^{\text{eff}}$ in this subclass of the Three Graces. In conclusion, the extra condition (46), added to Eqs. (28), now allows us to predict a small observed Λ_{eff} while keeping the Planck mass unrenormalized, so that the observed vacuum energy density $(M_{\text{Pl}}^{\text{bare}})^2 \Lambda_{\text{eff}}$ may be as small as wished.

Note that the six conditions (28) and (46) only need to be satisfied at one value of $X = X_c$. Therefore, there still remain six free functions, which do contribute to the evolution of the Universe before it reaches its equilibrium at $X = X_c$, as well as to the dynamics in generic non-symmetric situations or for perturbations around a spherically symmetric solution. However, the only physically relevant terms of the action, for our exact Schwarzschild-de Sitter background, are just a free $f_2(X)$ and $f_4 = k_4 X^{-5/2}$ and/or $s_4 = \kappa_4 X^{-3/2}$. All the other functions, including some non-trivial contributions to f_4 and s_4 which cancel in the combination (46), are passive for this solution, i.e., do not enter the result.

An example of a model satisfying all conditions (28) and (46) is given in Eqs. (21) above. Since both $f_4 = k_4 X^{-5/2}$ and $s_4 = \kappa_4 X^{-3/2}$ are allowed, it is also possible to use the Horndeski combination, such that $F_4 = 0$ in Eq. (6d). Then all field equations involve at most second derivatives, which simplifies their analysis (although the third derivatives of generalized Horndeski models with $F_4 \neq 0$ do not generate an extra degree of freedom, as recalled in Sec. II). In the present case, $F_4 = 0$ implies $k_4 = -6\kappa_4$, and this corresponds to $G_4(-M^2 X) = -2M^2 \kappa_4 X^{-1/2}$ in Eq. (6c). Let us choose $k_2 = \kappa_4 = -1$ to simplify. Then the specific model $f_2 = s_4 = -X^{-3/2}$ and $f_4 = 6X^{-5/2}$ is in the Horndeski class, and does not predict any renormalization of Newton's constant. It also predicts that the observed Hubble rate $H = M/(2\sqrt{6})$ is fully independent from the bare vacuum energy density $M_{\text{Pl}}^2 \Lambda_{\text{bare}}$ involved in action (3), and therefore does not change even after phase transitions possibly modifying this vacuum energy. On the other hand, this means that the Hubble scale H needs to be introduced by hand in the action via the mass scale M , therefore there still exist some fine-tuning in such a model, although it concerns the mass scale entering the action of a scalar field instead of the vacuum energy itself. A better model may be for instance $f_2 = -X^{-5/4}$, $f_4 = 6X^{-5/2}$ and $s_4 = -X^{-3/2}$, which is still in the Horndeski class and does not predict any renormalization of Newton's constant, but which now needs $M = (32M_{\text{Pl}}^2 \Lambda_{\text{bare}} H^2)^{1/6}$. In such a case, the mass scale M introduced in the action is thus intermediate between the huge Planck mass and the tiny Hubble rate.

VI. CONCLUSIONS

In this paper, we studied self-tuning in all shift-symmetric beyond Horndeski theories. Our goal is two-fold. First, we demonstrate that the theory does provide a mechanism to almost fully screen a very large bare cosmological constant entering the action, leaving a small effective (observable) one consistent with the present accelerated expansion of the Universe. Second, we select a subclass of beyond Horndeski theories which not only provide such a self-tuning of the cosmological constant, but also do not contradict Solar system tests.

Our starting point is the beyond Horndeski action (3) with only two mass scales in the action, the Planck mass M_{Pl} and an extra scale M . The theory contains six arbitrary functions, which specify the different possible kinetic terms of the scalar field, see Eqs. (4). We then progressively reduce the space of allowed models by imposing different physical requirements.

First we show that self-tuning is possible for a generic combination of beyond Horndeski Lagrangians, provided that the parameter M is adjusted to predict $\Lambda_{\text{eff}} \ll \Lambda_{\text{bare}}$. At this level

all the six functions of the theory are still allowed, the only constraint being on the magnitude of M — which may be either large or small with respect to $(M_{\text{Pl}}^2 \Lambda_{\text{bare}})^{1/4}$, depending on the model, but not of the same order of magnitude.

As a second step, we ask that the Schwarzschild-de Sitter (SdS) metric is a solution of the theory. This is a sufficient condition to satisfy (basic) Solar system tests of gravity. We find that an exact SdS solution does exist when the scalar field is such that $\varphi_\lambda \varphi^\lambda = \text{const.}$, provided the five conditions (28) are satisfied. Although the six functions still play a rôle before the Universe reaches this solution, as well as for the dynamics of perturbations around this solution, the conditions (28) effectively switch off three of them from the cosmological de Sitter evolution (and the SdS solution), making them passive (or “stealth”), so that the SdS solution does not feel them. The other three functions are f_2 , f_4 and s_4 , that we call the “Three Graces”. They are responsible for the resulting cosmological and SdS solution.

As a by-product of the above study, we found a class of regular black hole solutions, which can be considered as generalization of the self-tuning solutions found in [16, 24, 28]. Namely, beyond Horndeski theory satisfying conditions (31) at $X = q^2$, where $\dot{\varphi} = qM$ is the cosmological value of the scalar field time derivative, allows for self-tuning Schwarzschild-de Sitter black hole solutions with metric (32) and the non-trivial scalar field (34), (35).

Then we study perturbative corrections to the above solutions, allowing slightly non-SdS solutions. Doing so, we relax the above strict condition that the local solution must be of the exact Schwarzschild form. This allows us to take into account small deviations from GR which might not be observable with the present precision of local gravity tests. We find that in addition to the above Three Graces, the three other beyond Horndeski Lagrangians may give a small enough backreaction of the scalar field on the metric, notably when the local physics and the asymptotic cosmological behavior are not dominated by the same terms of the Lagrangian. On the other hand, when the same terms play a significant rôle both at small and large distances, the scalar backreaction is generically so large that Solar-system tests cannot be passed. There remains however one interesting subclass of models, satisfying condition (42), such that the deviations from the local Schwarzschild solution are small enough, even when the corresponding Lagrangians contribute significantly both at large and small distances. This condition (42) is a subset of the two (28c) we found when imposing an exact SdS solution.

It turns out, however, that when we take into account the renormalization of Newton’s constant G , which naturally happens for a time-dependent scalar field in the theory under consideration, the cosmological problem is *not* solved. This happens because the effective vacuum energy density has approximately the same value as the bare vacuum energy density, the two effects — effective decreasing of the cosmological constant and the effective increasing of the Planck mass — almost compensating each others, see Eq. (45). In order to solve the cosmological constant problem, while taking into account the renormalization of M_{Pl} , we need to impose the extra condition (46), in addition to (28). At this stage, we find that two out of the three functions entering the Three Graces must be very specific power laws, and there only remains one free function, $f_2(X)$, defining this subclass of allowed models.

To summarize, we found that the subclass of beyond Horndeski theory satisfying the six conditions (28) and (46) does solve the big cosmological constant problem, without any obvious contradiction with Solar system gravity tests.

More detailed analysis of Solar-system constraints is left for future work. Indeed we showed that we can choose the beyond Horndeski action such that the theory admits an exact SdS solution. However, this does not necessarily mean that all local gravity tests are

passed. Indeed, perturbations of planets (which are not included in our analysis) may give deviations from GR. For instance, the Nordtvedt effect, which tests the strong equivalence principle, would need to be studied in the present framework. It is tightly constrained by the *three*-body system Earth-Moon-Sun. The physics of the *interior* of stars may also be a way to additionally constrain these theories, notably because there exist couplings to the derivatives of the matter density in beyond Horndeski theories [42, 43].

Finally, the stability of the above SdS solutions is yet to be understood. We do know that some ghost or gradient instability exist in some models (for instance for $\mathcal{L}_{(2,0)} + \mathcal{L}_{(3,0)}$ in this self-tuning scenario), but this needs to be studied for the more promising Three Graces. We also leave this study for future work.

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Appendix A: Partial integration of the beyond-Horndeski Lagrangians

The Lagrangians (4) may be integrated by parts to be rewritten as follows:

$$\mathcal{L}_{(3,0)} = -M^2 \left[X f_3(X) + \frac{1}{2} \int f_3(X) dX \right] \square \varphi + \text{tot. div.}, \quad (\text{A1})$$

$$\begin{aligned} \mathcal{L}_{(4,0)} + \mathcal{L}_{(4,1)} &= -2M^2 X s_4(X) R \\ &\quad - \left[X f_4(X) + \int f_4(X) dX \right] \left[(\square \varphi)^2 - \varphi_{\mu\nu} \varphi^{\mu\nu} \right] \\ &\quad + \left[\int f_4(X) dX - 4s_4(X) \right] R^{\mu\nu} \varphi_\mu \varphi_\nu + \text{tot. div.}, \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \mathcal{L}_{(5,0)} + \mathcal{L}_{(5,1)} &= - \left[\frac{3}{2} \iint f_5(X) dX dX - 4X s_5(X) - 4 \int s_5(X) dX \right] G^{\mu\nu} \varphi_{\mu\nu} \\ &\quad - \frac{1}{2M^2} \left[2X f_5(X) + 3 \int f_5(X) dX \right] \left[(\square \varphi)^3 - 3 \square \varphi \varphi_{\mu\nu} \varphi^{\mu\nu} + 2 \varphi_{\mu\nu} \varphi^{\nu\rho} \varphi_\rho{}^\mu \right] \\ &\quad - \frac{1}{2M^2} \left[3 \int f_5(X) dX - 4s_5(X) \right] \\ &\quad \times \left[R \varphi^\mu \varphi_{\mu\nu} \varphi^\nu - 2 \square \varphi R^{\mu\nu} \varphi_\mu \varphi_\nu + 2 R^{\mu\nu\rho\sigma} \varphi_\mu \varphi_\rho \varphi_{\nu\sigma} \right] + \text{tot. div.} \end{aligned} \quad (\text{A3})$$

These expressions ease the translation of our notation (4) in terms of the functions G_n , F_n , A_n and B_n used in the literature, and explicitly given in Eqs. (6) and (7) above. Note that the first term of Eq. (A3) involves a double primitive of $f_5(X)$, i.e., a primitive of the single integral $\int f_5(X) dX$ entering other terms.

Appendix B: Field equations in a static and spherically symmetric situation

We give below the field equations of the most general shift-symmetric beyond Horndeski theory (3) when the metric is assumed to be static and spherically symmetric, in Schwarzschild coordinates (22), while imposing that the scalar field has the linear time dependence (23). These equations are used in Secs. IV and V, in which we first simplify them considerably by assuming $X \equiv -(\varphi_\lambda)^2/M^2 = \text{const.}$, and then linearize them around a flat metric for $|\varphi'| \ll |\dot{\varphi}|$.

Let us display first the time-time component of the Einstein equation (9a), globally multiplied by a factor $M^6 M_{\text{Pl}}^2 r^2$:

$$\begin{aligned}
& M^8 r^2 \left[M^2 q^2 (f_2 + 2X f_2') + e^{\nu-\lambda} \varphi'^2 f_2 \right] \\
& + \frac{1}{2} e^{-2\lambda} M^6 r (3f_3 + 2X f_3') \left[e^\nu r \varphi'^2 (\lambda' \varphi' - 2\varphi'') + e^\lambda M^2 q^2 (\varphi' (4 - r\lambda') + 2r\varphi'') \right] \\
& + e^{-3\lambda} \left[2e^\nu M^4 \varphi'^3 \{ \varphi' (r\lambda' (5f_4 + 2X f_4') - f_4) - 4r\varphi'' (2f_4 + X f_4') \} \right. \\
& \quad \left. - 2e^\lambda M^6 q^2 \varphi' \{ \varphi' [2X (r\lambda' - 1) f_4' + (7r\lambda' - 5) f_4] - 2r\varphi'' (5f_4 + 2X f_4') \} \right] \\
& + 3e^{-4\lambda} M^2 \varphi'^2 \left[e^\nu \{ \varphi'^2 (7\lambda' \varphi' - 10\varphi'') f_5 - 2e^\lambda M^2 X^2 (\lambda' \varphi' - 2\varphi'') f_5' \} \right. \\
& \quad \left. - e^\lambda M^2 q^2 (11\lambda' \varphi' - 14\varphi'') f_5 \right] \\
& + e^{-3\lambda} \left[e^\lambda \{ 4e^\lambda M^8 q^2 (e^\lambda + r\lambda' - 1) + 4e^\nu M^6 \varphi' (\varphi' (e^\lambda - 3r\lambda' + 1) + 4r\varphi'') \} s_4 \right. \\
& \quad + 8M^4 \{ M^4 q^4 e^{2\lambda-\nu} (e^\lambda + r\lambda' - 1) - e^\lambda (e^\lambda - 1) M^2 q^2 \varphi'^2 \\
& \quad \left. + e^\nu r \varphi'^3 (\lambda' \varphi' - 2\varphi'') \} s_4' \right] \\
& + e^{-4\lambda} \left[e^\lambda \{ 6e^\nu M^4 \varphi'^2 ((e^\lambda - 5) \lambda' \varphi' - 2(e^\lambda - 3) \varphi'') \right. \\
& \quad - 6e^\lambda M^6 q^2 ((e^\lambda - 3) \lambda' \varphi' - 2(e^\lambda - 1) \varphi'') \} s_5 \\
& \quad + e^{-\nu} \{ -4e^{2\lambda} M^6 q^4 ((e^\lambda - 3) \lambda' \varphi' - 2(e^\lambda - 1) \varphi'') \\
& \quad + 8(e^\lambda - 1) M^4 q^2 e^{\lambda+\nu} \varphi'^2 (\lambda' \varphi' - 2\varphi'') \\
& \quad \left. - 4(e^\lambda - 3) e^{2\nu} M^2 \varphi'^4 (\lambda' \varphi' - 2\varphi'') \} s_5' \right] \\
& = M^6 M_{\text{Pl}}^2 e^{\nu-\lambda} [1 - r\lambda' - e^\lambda (1 - \Lambda_{\text{bare}} r^2)], \tag{B1}
\end{aligned}$$

where $X = e^{-\nu} q^2 - e^{-\lambda} \varphi'^2/M^2$, and where the primes denote derivatives with respect to the argument of the corresponding terms, i.e., $f'_n = df_n(X)/dX$ and $s'_n = ds_n(X)/dX$, but $\varphi' = \partial_r \varphi$, $\varphi'' = \partial_r^2 \varphi$, $\lambda' = \partial_r \lambda$ and $\nu' = \partial_r \nu$.

The second equation expresses that the linear combination (10) vanishes for $\mu = \nu = r$,

and we multiply it by a global factor $e^{2\lambda} M^4 M_{\text{Pl}}^2 r^2$:

$$\begin{aligned}
& e^\lambda M^8 r^2 X f_2 \\
& + e^{-2\lambda-\nu} [2e^\nu M^2 \varphi'^4 (r\nu' + 1) - 2e^\lambda M^4 q^2 \varphi' (\varphi' (-r\lambda' + 2r\nu' + 1) + 2r\varphi'')] f_4 \\
& + 6e^{-3\lambda-\nu} \varphi'^2 [e^\nu \nu' \varphi'^3 + e^\lambda M^2 q^2 (\varphi' (\lambda' - 2\nu') - 2\varphi'')] f_5 \\
& + [4e^{-\nu} M^6 q^2 (e^\lambda + r\nu' - 1) - 4e^{-\lambda} M^4 \varphi'^2 (e^\lambda + r\nu' + 1)] s_4 \\
& + 8M^4 q^2 r e^{-\lambda-2\nu} [e^\lambda M^2 q^2 \nu' - e^\nu \varphi' (\lambda' \varphi' - 2\varphi'')] s'_4 \\
& + 4M^2 e^{-2(\lambda+\nu)} \varphi' [e^\lambda M^2 \nu' (3e^{2\nu} X s_5 + 2q^4 s'_5) - 2e^\nu q^2 \varphi' (\lambda' \varphi' - 2\varphi'')] s'_5] \\
& = -M^4 M_{\text{Pl}}^2 [1 + r\nu' - e^\lambda (1 - \Lambda_{\text{bare}} r^2)] . \tag{B2}
\end{aligned}$$

Note that no derivative of any function f_n enters this linear combination (10), although some s'_4 and s'_5 do remain, as underlined at the end of Sec. II. Note in particular that the function f_3 fully disappears from this combination. The reason is that the same term $\propto (3f_3 + 2X f'_3)$ enters both the rr -component of the Einstein equations and the scalar current, and we know that f'_3 must cancel in the combination (10).

The third equation is the radial component of the scalar current (8b), globally multiplied by a factor $-2e^\lambda M^6 r^2$:

$$\begin{aligned}
& 4M^8 r^2 \varphi' (f_2 + X f'_2) \\
& + M^6 r e^{-\lambda-\nu} (3f_3 + 2X f'_3) [e^\nu \varphi'^2 (r\nu' + 4) - e^\lambda M^2 q^2 r\nu'] \\
& + e^{-2\lambda-\nu} [8e^\nu M^4 \varphi'^3 (r\nu' + 1) (2f_4 + X f'_4) - 4e^\lambda M^6 q^2 r \varphi' (\nu' (5f_4 + 2X f'_4) + \lambda' f_4)] \\
& - 6M^2 e^{-3\lambda-\nu} \varphi'^2 [e^\lambda M^2 q^2 (2\lambda' + 7\nu') f_5 - e^\nu \nu' (5\varphi'^2 f_5 - 2e^\lambda M^2 X^2 f'_5)] \\
& + 16M^4 e^{-2\lambda-\nu} \varphi' [e^\lambda M^2 q^2 (e^\lambda + r\lambda' - 1) s'_4 + e^\nu (e^\lambda - r\nu' - 1) (e^\lambda M^2 s_4 - \varphi'^2 s'_4)] \\
& + e^{-3\lambda-2\nu} [-8e^{2\lambda} (e^\lambda - 1) M^6 q^4 \nu' s'_5 + 4 (e^\lambda - 3) e^{2\nu} M^2 \nu' \varphi'^2 (3e^\lambda M^2 s_5 - 2\varphi'^2 s'_5) \\
& \quad - 4M^4 q^2 e^{\lambda+\nu} \{ (e^\lambda - 1) \nu' (3e^\lambda M^2 s_5 - 4\varphi'^2 s'_5) - 4\lambda' \varphi'^2 s'_5 \}] = 0. \tag{B3}
\end{aligned}$$

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